第十二章　多元多项式

§1　多元多项式

1. 设 $f(x_1,\cdots,x_n)$ 是数域 K 上的 n 元齐次多项式.

证明: 如果存在数域 K 上的 n 元多项式 $g(x_1,\cdots,x_n)$ 与 $h(x_1,\cdots,x_n)$, 使

$$f(x_1,\cdots,x_n)=g(x_1,\cdots,x_n)h(x_1,\cdots,x_n),$$

则 $g(x_1,\cdots,x_n)$ 与 $h(x_1,\cdots,x_n)$ 也都是齐次多项式.

证明: 设 $\deg f=m$, $\deg g=k$, $\deg h=l$. 令

$$g=g_p+g_{p+1}+\cdots+g_k,\qquad h=h_q+h_{q+1}\cdots+h_l,$$

其中 g_i,h_j 分别为 i,j 次齐次多项式, 且 g_p,h_q 是分解中次数最低的齐次多项式, $k+l=m$, 则

$$f=g_ph_q+\sum_{t=p+q+1}^{m}\left(\sum_{i+j=t}g_ih_j\right).$$

因此当 $p+q<m$ 时 f 不是齐次多项式. 而 $p+q=k+l=m$ 可推出 $p=k$, $q=l$, 因此 $g=g_k,h=h_l$ 都是齐次多项式.

2. 设 $f(x,y)\in K[x,y]$. 证明: 如果 $f(x,x)=0$, 则 $x-y\mid f(x,y)$.

证明: 设 $f(x,y)=\sum\limits_{k=0}^{n}a_k(x)y^k$, 则

$$\begin{aligned}f(x,y)&=f(x,y)-f(x,x)=\sum_{k=0}^{n}a_k(x)(y^k-x^k)\\&=(y-x)\sum_{k=1}^{n}a_k(x)(y^{k-1}+y^{k-2}x+\cdots+yx^{k-2}+x^{k-1}).\end{aligned}$$

因此 $x-y\mid f(x,y)$.

*3. 计算下列行列式:

$$\begin{vmatrix} \dfrac{1}{x_1-a_1} & \dfrac{1}{x_1-a_2} & \cdots & \dfrac{1}{x_1-a_n} \\ \dfrac{1}{x_2-a_1} & \dfrac{1}{x_2-a_2} & \cdots & \dfrac{1}{x_2-a_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{x_n-a_1} & \dfrac{1}{x_n-a_2} & \cdots & \dfrac{1}{x_n-a_n} \end{vmatrix}.$$

解: 把原行列式记为 $D_n(x_1,\cdots,x_n,a_1,\cdots,a_n)$. 则

$$D_n(x_1,\cdots,x_n,a_1,\cdots,a_n)=\frac{G(x_1,\cdots,x_n,a_1,\cdots,a_n)}{F(x_1,\cdots,x_n,a_1,\cdots,a_n)},$$

其中 G 与 F 都是 $x_1,\cdots,x_n,a_1,\cdots,a_n$ 的多项式. 易知

$$F(x_1,\cdots,x_n,a_1,\cdots,a_n)=\prod_{1\le i,j\le n}(x_i-a_j).$$

由于

$$D_n(x_1,\cdots,x_i,\cdots,x_i,\cdots,x_n,a_1,\cdots,a_n)=0,$$
$$D_n(x_1,\cdots,x_n,a_1,\cdots,a_i,\cdots,a_i,\cdots,a_n)=0,$$

可得

$$\begin{aligned}&G(x_1,\cdots,x_n,a_1,\cdots,a_n)\\ &=\prod_{1\le i<j\le n}(x_i-x_j)\prod_{1\le i<j\le n}(a_j-a_i)\cdot G_1(x_1,\cdots,x_n,a_1,\cdots,a_n).\end{aligned}$$

比较两边 x_i 与 a_j 的次数 (都是 $n-1$ 次), 可知 $G_1=c_n$ 是一个常数. 因此

$$D_n(x_1,\cdots,x_n,a_1,\cdots,a_n)=\frac{\prod\limits_{1\le i<j\le n}(x_i-x_j)\prod\limits_{1\le i<j\le n}(a_j-a_i)\cdot c_n}{\prod\limits_{1\le i,j\le n}(x_i-a_j)}.$$

又因

$$\begin{aligned}&[(x_n-a_n)D_n(x_1,\cdots,x_n,a_1,\cdots,a_n)]_{x_n=a_n}\\ &=D_{n-1}(x_1,\cdots,x_{n-1},a_1,\cdots,a_{n-1}),\end{aligned}$$

所以

$$\frac{\left(\prod\limits_{1\le i<j\le n-1}(x_i-x_j)\right)(x_1-a_n)\cdots(x_{n-1}-a_n)\left(\prod\limits_{1\le i<j\le n-1}(a_j-a_i)\right)(a_n-a_1)\cdots(a_n-a_{n-1})c_n}{\left(\prod\limits_{1\le i,j\le n-1}(x_i-a_j)\right)(a_n-a_1)\cdots(a_n-a_{n-1})(x_1-a_n)\cdots(x_{n-1}-a_n)}$$

$$= \frac{\prod\limits_{1\le i<j\le n-1}(x_i-x_j)\prod\limits_{1\le i<j\le n-1}(a_j-a_i)\cdot c_n}{\prod\limits_{1\le i,j\le n-1}(x_i-a_j)}$$

$$= D_{n-1}(x_1,\cdots,x_{n-1},a_1,\cdots,a_{n-1}),$$

可得 $c_n=c_{n-1}$. 依此类推, 最终可得 $c_n=c_1=1$. 因而

$$D_n(x_1,\cdots,x_n,a_1,\cdots,a_n)=\frac{\prod\limits_{1\le i<j\le n}(x_i-x_j)(a_j-a_i)}{\prod\limits_{1\le i,j\le n}(x_i-a_j)}.$$

§2 对称多项式

1. 用初等对称多项式表示下列对称多项式:

(1) $x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2$;

(2) $x_1^2x_2^2+x_1^2x_3^2+x_1^2x_4^2+x_2^2x_3^2+x_2^2x_4^2+x_3^2x_4^2$;

(3) $(x_1+x_2)(x_1+x_3)(x_2+x_3)$;

(4) $(x_1x_2+x_3)(x_1x_3+x_2)(x_2x_3+x_1)$;

(5) $(x_1^2+x_2^2)(x_1^2+x_3^2)(x_2^2+x_3^2)$;

(6) $(x_1+x_2+x_1x_2)(x_2+x_3+x_2x_3)(x_1+x_3+x_1x_3)$.

解: (1) 原式 $=x_1x_2(x_1+x_2+x_3)+x_1x_3(x_1+x_2+x_3)+x_2x_3(x_1+x_2+x_3)-3x_1x_2x_3=\sigma_1\sigma_2-3\sigma_3$.

(2)

2	2	0	0	σ_2^2
2	1	1	0	$\sigma_1\sigma_3$
1	1	1	1	σ_4

因此原式 $=\sigma_2^2+A\sigma_1\sigma_3+B\sigma_4$.

取 $x_1=x_2=x_3=1$, $x_4=0$, 得 $3=9+3A$, $A=-2$;

取 $x_1=x_2=x_3=x_4=1$, 得 $B=2$;

故原式 $=\sigma_2^2-2\sigma_1\sigma_3+2\sigma_4$.

(3) 原式 $=(\sigma_1-x_3)(\sigma_1-x_2)(\sigma_1-x_1)=\sigma_1^3-\sigma_1\sigma_1^2+\sigma_2\sigma_1-\sigma_3=\sigma_1\sigma_2-\sigma_3$.

(4) 原式 $=\dfrac{1}{\sigma_3}(\sigma_3+x_3^2)(\sigma_3+x_2^2)(\sigma_3+x_1^2)$. 由于

$$x_1^2+x_2^2+x_3^2=\sigma_1^2-2\sigma_2,$$

$$x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2=\sigma_2^2-2\sigma_1\sigma_3,$$

$$x_1^2x_2^2x_3^2 = \sigma_3^2,$$

$$\begin{aligned}\text{原式} &= \frac{1}{\sigma_3}(\sigma_3^3 + (\sigma_1 - 2\sigma_2)\sigma_3^2 + (\sigma_2^2 - \sigma_1\sigma_3)\sigma_3 + \sigma_3^2) \\ &= \sigma_1^2\sigma_3 - 2\sigma_1\sigma_3 + \sigma_2^2 - 2\sigma_2\sigma_3 + \sigma_3^2 + \sigma_3.\end{aligned}$$

(5) 原式$= (x_1^2 + x_2^2 + x_3^2 - x_3^2)(x_1^2 + x_2^2 + x_3^2 - x_2^2)(x_1^2 + x_2^2 + x_3^2 - x_1^2)$
$= (\sigma_1^2 - 2\sigma_2 - x_3^2)(\sigma_1^2 - 2\sigma_2 - x_2^2)(\sigma_1^2 - 2\sigma_2 - x_1^2)$
$= (\sigma_1^2 - 2\sigma_2 - x_3^2)^3 - (\sigma_1^2 - 2\sigma_2 - x_3^2)(\sigma_1^2 - 2\sigma_2 - x_3^2)^2 + (\sigma_1^2 - 2\sigma_1\sigma_3)(\sigma_1^2 - 2\sigma$
$= \sigma_1^2\sigma_2^2 - 2\sigma_1^3\sigma_3 - 2\sigma_2^3 + 4\sigma_1\sigma_2\sigma_3 - \sigma_3^2$

(6) 原式$= \dfrac{1}{\sigma_3}[(\sigma_3 + \sigma_2)^3 - \sigma_2(\sigma_3 + \sigma_2)^2 + \sigma_1\sigma_3(\sigma_3 + \sigma_2) - \sigma_3^2]$
$= \dfrac{1}{\sigma_3}[\sigma_3(\sigma_3 + \sigma_2)^2 + \sigma_1\sigma_3(\sigma_3 + \sigma_2) - \sigma_3^2]$
$= \sigma_2^2 + 2\sigma_2\sigma_3 + \sigma_3^2 + \sigma_1\sigma_3 + \sigma_1\sigma_2 - \sigma_3^2.$

2. 用初等对称多项式表示下列 n 元对称多项式:

(1) $\sum x_1^4$;　　(2) $\sum x_1^2x_2^2$;

(3) $\sum x_1^2x_2x_3$;　　(4) $\sum x_1^2x_2^2x_3x_4$.

解: (1) $\sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_2 - 4\sigma_4$.

(2) $\sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4$.

(3) $\sigma_1\sigma_3 - 4\sigma_4$.

(4) $\sigma_2\sigma_4 - 4\sigma_1\sigma_5 + 9\sigma_6$.

3. 设 x_1, x_2, x_3 是方程 $3x^3 - 5x^2 + 1$ 的三个根. 计算

$$x_1^3x_2 + x_1x_2^3 + x_1^3x_3 + x_1x_3^3 + x_2^3x_3 + x_2x_3^3.$$

解: 原式 $= \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 = \dfrac{5}{9}$.

4. 设 $xyz \neq 0$, 且 $x + y + z = 0$, 求:

$$\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}.$$

解: 原式$= \dfrac{x^3 + y^3 + z^3}{xyz}$
$= \dfrac{\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3}{xyz} = 3.$

5. 证明: 三次方程 $x^3 + a_1x^2 + a_2x + a_3 = 0$ 的三个根成等差数列的充分必要条件是

$$2a_1^3 - 9a_1a_2 + 27a_3 = 0.$$

证明: 三个根成等差数列的充分必要条件是以下 3 个数

$$x_1+x_2-2x_3,\quad x_1+x_3-2x_2,\quad x_2+x_3-2x_1,$$

中至少有一个等于 0. 故

$$\text{三个根成等差数列} \iff (x_1+x_2-2x_3)(x_1+x_3-2x_2)(x_2+x_3-2x_1)=0.$$

而

$$\begin{aligned}&(x_1+x_2-2x_3)(x_1+x_3-2x_2)(x_2+x_3-2x_1)\\&=(x_1+x_2+x_3-3x_3)(x_1+x_2+x_3-3x_2)(x_1+x_2+x_3-3x_1)\\&=(-a_1)^3-3(-a_1)(-a_1)^2+9a_2(-a_1)-27(-a_3)\\&=2a_1^3-9a_1a_2+27a_3.\end{aligned}$$

*__6.__ 设 $x_1,x_2,\cdots,x_n$ 是方程

$$x^n+a_1x^{n-1}+\cdots+a_n=0$$

的根, 证明: $x_2,x_3,\cdots,x_n$ 的对称多项式可表成 x_1 与 $a_1,a_2,\cdots,a_n$ 的多项式.

证明: 设

$$f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)=\sum_{k=0}^{n}(-1)^k a_k x^{n-k}.$$

从而

$$\begin{aligned}(x-x_2)\cdots(x-x_n)&=\frac{f(x)}{x-x_1}=\frac{f(x)-f(x_1)}{x-x_1}\\&=\frac{\sum\limits_{k=0}^{n}(-1)^k a_k x^{n-k}-\sum\limits_{k=0}^{n}(-1)^k a_k x_1^{n-k}}{x-x_1}\\&=\sum_{k=0}^{n-1}(-1)^k a_k(x^{n-k-1}+x^{n-k-2}x_1+\cdots+x^{n-n-1}).\end{aligned}$$

由最后一式知 x 的各次项系数都是 x_1 与 $a_1,\cdots,a_n$ 的多项式 $(a_0=1)$, 从而 $x_2,\cdots,x_n$ 的初等对称多项式是 x_1 与 $a_1,\cdots,a_n$ 的多项式, 进而由对称多项式基本定理知 $x_2,\cdots,x_n$ 的对称多项式可表成是 x_1 与 $a_1,\cdots,a_n$ 的多项式.

*__7.__ 设

$$f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$$

$$= x^n - \sigma_1 x^{n-1} + \cdots + (-1)^n \sigma_n,$$
$$s_k = x_1^k + x_2^k + \cdots + x_n^k, \quad (k = 0, 1, 2, \cdots).$$

(1) 证明:

$$x^{k+1} f'(x) = (s_0 x^k + s_1 x^{k-1} + \cdots + s_{k-1} x + s_k) f(x) + g(x),$$

其中 $g(x)$ 的次数 $< n$ 或 $g(x) = 0$.

(2) 证明**牛顿 (Newton) 公式**:

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k k\sigma_k = 0 \quad k \leqslant n,$$
$$s_k - \sigma_1 s_{k-1} + \cdots + (-1)^n \sigma_n s_{k-n} = 0 \quad k > n.$$

证明: 设 $g(x) = \sum\limits_{i=1}^{n} \dfrac{x_i^{k+1} f(x)}{x - x_i}$, 则 $g(x) = 0$ 或 $\deg g(x) < n$. 而

$$\begin{aligned} x^{k+1} f'(x) - g(x) &= \sum_{i=1}^{n} \frac{x^{k+1} f(x)}{x - x_i} - \sum_{i=1}^{n} \frac{x_i^{k+1} f(x)}{x - x_i} = \left(\frac{x^{k+1} - x_i^{k+1}}{x - x_i} \right) f(x) \\ &= \sum_{i=1}^{n} \sum_{j=0}^{k} (x^{k-j} x_i^j f(x) = \sum_{j=0}^{k} \left(\sum_{i=1}^{n} (x^{k-j} x_i^j \right) f(x) \\ &= \left(\sum_{j=0}^{k} x^{k-j} s_j \right) f(x) \\ &= (s_0 x^k + s_1 x^{k-1} + \cdots + s_{k-1} x + s_k) f(x). \end{aligned}$$

即得所证.

(2) 比较等式

$$x^{k+1} f'(x) = (s_0 x^k + s_1 x^{k-1} + \cdots + s_{k-1} x + s_k) f(x) + g(x)$$

两边 n 次项系数, 由于 $g(x)$ 的次数 $< n$ 或 $g(x) = 0$, 所以

$x^{k+1} f'(x)$ 的n次项系数=$(s_0 x^k + s_1 x^{k-1} + \cdots + s_{k-1} x + s_k) f(x)$ 的n次项系数,

所以当 $k \leq n$ 时,

$$(n-k)(-1)^k \sigma_k = s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \cdots + (-1)^k \sigma_k s_0,$$

即

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k k\sigma_k = 0.$$

当 $k > n$,

$$0 = s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \cdots + (-1)^n \sigma_n s_{k-n},$$

即得所证.

*__8.__ 用初等对称多项式表示 s_2, s_3, s_4, s_5.

解: $s_2 = \sigma_1^2 - 2\sigma_2$,

$s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$,

$s_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4$,

$s_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$.

*§3 结式

1. 计算下列多项式的结式:

(1) $f(x) = x^3 - 3x^2 + 2x + 1$, $g(x) = 2x^2 - x - 1$;

(2) $f(x) = 2x^3 - 3x^2 - x + 2$, $g(x) = x^4 - 2x^3 - 3x + 4$;

解: (1) $\operatorname{Res}(f, g) = (-1)^{2\cdot 3}\operatorname{Res}(2x^2 - x - 1, f) = (-1)^6 \cdot 2^3 \cdot f\left(-\frac{1}{2}\right) f(1) = -7$.

(2) $f(x), g(x)$ 有公共根 1, 所以结式 $\operatorname{Res}(f, g) = 0$.

2. 当 λ 取何值时, 下列多项式有公共根:

(1) $f(x) = x^3 - \lambda x + 2$, $g(x) = x^2 + \lambda x + 2$;

(2) $f(x) = x^3 + \lambda x^2 - 9$, $g(x) = x^3 + \lambda x - 3$.

解: (1) $\operatorname{Res}(f, g) = -4(\lambda + 1)^2(\lambda - 3)$, 故当 $\lambda = -1$ 或 3 时有公共根.

(2) $\operatorname{Res}(f, g) = 9(\lambda^2 + 12)(\lambda^2 + 2)$, 故当 $\lambda = \pm 2\sqrt{3}\mathrm{i}$ 或 $\pm\sqrt{2}\mathrm{i}$ 时有公共根.

3. 求下列曲线的直角坐标方程:

(1) $x = t^2 + t - 1$, $y = 2t^2 + t - 1$;

(2) $x = \dfrac{t - 1}{t^2 + 1}$, $y = \dfrac{t^2 + t - 1}{t^2 + 1}$.

解: (1) $4x^2 - 4xy + y^2 + 5x - 3y + 1 = 0$.

(2) $5x^2 - 6xy + 2y^2 + 5x - 3y + 1 = 0$.

4. 当 λ 为何值时, 下列多项式有重根?

(1) $f(x) = x^3 - 3x + \lambda$; (2) $f(x) = x^4 - 4x^3 + (2 - \lambda)x^2 + 2x - 2$.

解: (1) $2, -2$;

(2) $-1, -\dfrac{3}{2}, \dfrac{7}{2} + \dfrac{9}{2}\sqrt{3}\mathrm{i}, \dfrac{7}{2} - \dfrac{9}{2}\sqrt{3}\mathrm{i}$.

5. 求下列方程组的解:

(1) $\begin{cases} 5x^2 - 6xy + 5y^2 = 16, \\ 2x^2 - xy + y^2 - x - y = 4; \end{cases}$ (2) $\begin{cases} x^2 + y^2 + 4x - 2y = -3, \\ x^2 + 4xy - y^2 + 10y = 9. \end{cases}$

解: (1) $\operatorname{Res}_y(f, g) = 32(y^4 - y^3 - 3y^2 + y + 2)$,

$$\begin{cases} x = 1 \\ y = -1 \end{cases} \qquad \begin{cases} x = -1 \\ y = 1 \end{cases} \qquad \begin{cases} x = 2 \\ y = 2 \end{cases}$$

(2) $\operatorname{Res}_x(f, g) = 4(5x^4 + 40x^3 + 106x^2 + 104x + 33)$,

$$\begin{cases} x = -1 \\ y = 2 \end{cases} \qquad \begin{cases} x = -3 \\ y = 0 \end{cases} \qquad \begin{cases} x = -2 + \frac{3}{5}\sqrt{5} \\ y = 1 + \frac{1}{5}\sqrt{5} \end{cases} \qquad \begin{cases} x = -2 - \frac{3}{5}\sqrt{5} \\ y = 1 - \frac{1}{5}\sqrt{5} \end{cases}$$

6. 求下列圆锥曲线的交点坐标:

(1) 圆 $x^2 + y^2 - 3x - y = 0$ 与双曲线 $x^2 + 2xy - y^2 - 4y - 2 = 0$;

(2) 双曲线 $4x^2 - 7xy + y^2 + 13x - 2y - 3 = 0$ 与双曲线 $9x^2 - 14xy + y^2 + 28x - 4y - 5 = 0$.

解: (1) $(1, -1), \left(\frac{3}{2} + \frac{1}{2}\sqrt{2}, \frac{1}{2} + \sqrt{2}\right), \left(\frac{3}{2} - \frac{1}{2}\sqrt{2}, \frac{1}{2} - \sqrt{2}\right)$;

(2) $(0, -1), (1, 2), (2, 3), (-2, 1)$.

7. 证明结式的下列性质: 设 $f(x), g(x)$ 分别是 n 次与 m 次多项式. 则

(1) $\operatorname{Res}(f, g) = (-1)^{mn} \operatorname{Res}(g, f)$;

(2) $\operatorname{Res}(af, bg) = a^m b^n \operatorname{Res}(f, g)$;

*(3) $\operatorname{Res}((x - a)f, g) = g(a) \operatorname{Res}(f, g)$.

证明: (1), (2) 显然. 今证 (3). 设

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \qquad g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m,$$

则

$$(x - a)f(x) = a_0x^{n+1} + (a_1 - a_0a)x^n + \cdots + (a_n - a_{n-1}a)x - a_na.$$

$$\operatorname{Res}((x-a)f,g)=$$

$$\begin{vmatrix} a_0 & a_1-a_0a & a_2-a_1a & \cdots & a_n-a_{n-1}a & -a_na & & \\ & a_0 & a_1-a_0a & a_2-a_1a & \cdots & a_n-a_{n-1}a & -a_na & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \\ & & a_0 & \cdots & \cdots & \cdots & a_n-a_{n-1}a & -a_na \\ b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & & \\ & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \\ & & b_0 & \cdots & \cdots & \cdots & b_{m-1} & b_m \end{vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m+1 \end{matrix}$$

自第一列起, 各列乘 a 加到后一列, 直至最后一列, 可得

$$\operatorname{Res}((x-a)f,g)=$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & & \\ & a_0 & a_1 & a_2 & \cdots & a_n & 0 & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \\ & & a_0 & \cdots & \cdots & \cdots & a_n & 0 \\ b_0 & b_1+b_0a & b_2+b_1a+b_0a^2 & \cdots & \cdots & g(a) & \cdots & g(a)a^m \\ & b_0 & b_1+b_0a & \cdots & \cdots & \cdots & \cdots & g(a)a^{m-1} \\ & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ & & b_0 & \cdots & \cdots & \cdots & \cdots & g(a) \end{vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m+1 \end{matrix}$$

从最后一行起, 各行乘 $(-a)$ 加到前一行, 直到第 $n+1$ 行, 再按最后一列展开, 可得

$$\operatorname{Res}((x-a)f,g)=g(a)\operatorname{Res}(f,g).$$

*8. 设 $f(x)=a(x-x_1)\cdots(x-x_n)$, $g(x)=b(x-y_1)\cdots(x-y_m)$.

证明: $\operatorname{Res}(f,g)=a^m\prod\limits_{i=1}^{n}g(x_i)=(-1)^{mn}b^n\prod\limits_{j=1}^{m}f(y_j)=a^mb^n\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{m}(x_i-y_j)$.

证明:
$$\begin{aligned}\operatorname{Res}(f,g)&=a^m\operatorname{Res}((x-x_1)\cdots(x-x_n),g(x))\\&=a^mg(x_1)\operatorname{Res}((x-x_2)\cdots(x-x_n),g(x))\\&=a^mg(x_1)g(x_2)\cdots g(x_n)\\&=a^mb^n\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i-y_j)=(-1)^{mn}b^n\prod_{j=1}^{m}f(y_j).\end{aligned}$$

*9. 证明: $\operatorname{Res}(f(x),g_1(x)g_2(x))=\operatorname{Res}(f(x),g_1(x))\operatorname{Res}(f(x),g_2(x))$.

证明: 设

$$f(x) = a(x-x_1)\cdots(x-x_n),$$

则

$$\begin{aligned}\operatorname{Res}(f, g_1g_2) &= a^{\deg g_1g_2}\prod_{i=1}^{n} g_1(x_i)g_2(x_i)\\ &= a^{\deg g_1}\prod_{i=1}^{n} g_1(x_i)a^{\deg g_2}\prod_{i=1}^{n} g_2(x_i)\\ &= \operatorname{Res}(f, g_1)\operatorname{Res}(f, g_2).\end{aligned}$$

*10. 设 f 为首一多项式, 证明: 对任意多项式 h, $\operatorname{Res}(f,g) = \operatorname{Res}(f, g+hf)$.

证明: 设 $f(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$, 则

$$\begin{aligned}\operatorname{Res}(f, g+hf) &= \prod_{i=1}^{n}(g(x_i)+h(x_i)f(x_i))\\ &= \prod_{i=1}^{n} g(x_i) = \operatorname{Res}(f,g).\end{aligned}$$

*11. 利用习题 7 至 10 证明的结式性质计算下列多项式的结式:

(1) $f(x) = x^n + x + 1$, $g(x) = x^2 - 3x + 2$;

(2) $f(x) = x^n + 1$, $g(x) = (x-1)^n$;

(3) $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$,
$g(x) = a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-2}x + a_{n-1}$.

(4) $f(x) = \dfrac{x^n-1}{x-1}$, $g(x) = \dfrac{x^m-1}{x-1}$;

解: (1) $\operatorname{Res}(f,g) = (-1)^{2n}(1+1+1)(2^n+2+1) = 3(2^n+3)$.

(2) $\operatorname{Res}(f,g) = (-1)^n \cdot 2^n$.

(3) 由于 $f(x) = xg(x) + a_n$, 所以

$$\operatorname{Res}(f,g) = (-1)^{n(n-1)}\operatorname{Res}(g,f) = (-1)^{n(n-1)}\operatorname{Res}(g,a_n) = a_n^{n-1}.$$

(4) (a) 如 $(m,n) = d > 1$, 则 $\dfrac{x^n-1}{x-1}$ 与 $\dfrac{x^m-1}{x-1}$ 有公共根, 因此 $\operatorname{Res}(f,g) = 0$.

(b) 如 $(m,n) = 1$, 不妨设 $n > m$, 则 $n = mq + r$, $0 \le r < m$. 显然 $(m,r) = 1$. 则

$$\frac{x^n-1}{x-1} = \frac{x^{mq}x^r-1}{x-1} = \frac{(x^{mq}-1)x^r + x^r - 1}{x-1},$$

从而

$$\operatorname{Res}\left(\frac{x^n-1}{x-1}, \frac{x^m-1}{x-1}\right) = (-1)^{m-1)(n+r)}\operatorname{Res}\left(\frac{x^r-1}{x-1}, \frac{x^m-1}{x-1}\right).$$

我们证明 $(m-1)(n+r)$ 一定是偶数.

如 $m-1$ 是偶数, 则结论成立. 现设 $m-1$ 是奇数, 则 m 为偶数, 从而 n 是奇数, r 也是奇数, 于是 $n+r$ 是偶数. 从而

$$\operatorname{Res}\left(\frac{x^n-1}{x-1}, \frac{x^m-1}{x-1}\right) = \operatorname{Res}\left(\frac{x^r-1}{x-1}, \frac{x^m-1}{x-1}\right).$$

再用 r 除 m, 根据辗转相除法的原理, 由 $(m,r)=1$ 可得

$$\operatorname{Res}\left(\frac{x^r-1}{x-1}, \frac{x^m-1}{x-1}\right) = \cdots = \operatorname{Res}\left(\frac{x^{r'}-1}{x-1}, 1\right) = 1.$$

即当 $(m,n)=1$ 时 $\operatorname{Res}\left(\frac{x^n-1}{x-1}, \frac{x^m-1}{x-1}\right) = 1$.

*$\mathbf{12.}$ 设 $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \in K[x]$,

证明: $f(x)$ 的判别式

$$D(f) = (-1)^{\frac{n(n-1)}{2}} a_0^{-1}\operatorname{Res}(f, f').$$

证明:

$$\begin{aligned}
D(f) &= a_0^{2n-2}\prod_{1\le i<j\le n}(x_i-x_j)^2 = (-1)^{\frac{n(n-1)}{2}} a_0^{2n-2}\prod_{i\ne j}(x_i-x_j) \\
&= (-1)^{\frac{n(n-1)}{2}} a_0^{2n-2}\prod_{i=1}^{n}(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n) \\
&= (-1)^{\frac{n(n-1)}{2}} a_0^{n-2}\prod_{i=1}^{n} f'(x_i) \\
&= (-1)^{\frac{n(n-1)}{2}} a_0^{n-2}\operatorname{Res}((x-x_1)\cdots(x-x_n), f') \\
&= (-1)^{\frac{n(n-1)}{2}} a_0^{-1}\operatorname{Res}(f, f').
\end{aligned}$$

*§4 吴消元法

1. 仿照例 4.4 分别用分步法及一步法解多项式方程组:

$$\begin{cases}
-12x_2^2 + 7x_1x_2 - 2 = 0, \\
-2x_3 + x_1^2 = 0, \\
-x_3^2 + x_1x_2 + 2 = 0.
\end{cases}$$

解: 这里只列出分步法的过程, 并列出部分运算结果.

```
>read `d:/mapleuser/wsolve2`:
>P1:=-12*x2^2+7*x1*x2-2;
>P2:=-2*x3+x1^2;
>P3:=-x3^2+x1*x2+2;
>PS1:={P1,P2,P3};
>ord:=[x3,x2,x1];
>B1:=basset(PS1,ord);
```

$$B_1 := [-2x_3 + x_1^2, -12x_2^2 + 7x_1x_2 - 2]$$

```
>R1:=remseta(PS1,B1,ord);
```

$$R_1 := \{8 + 4x_1x_2 - x_1^4\}$$

```
>PS2:={op(B1)} union R1;
>B2:=basset(PS2,ord);
```

$$B_2 := [-2x_3 + x_1^2, 8 + 4x_1x_2 - x_1^4]$$

```
>R2:=remseta(PS2,B2,ord);
```

$$R_2 := \{64x_1^2 + 192 - 48x_1^4 - 7x_1^6 + 3x_1^8\}$$

```
>PS3:={op(B2)} union R2;
>B3:=basset(PS3,ord);
```

$$B_3 := [-2x_3 + x_1^2, 8 + 4x_1x_2 - x_1^4, 64x_1^2 + 192 - 48x_1^4 - 7x_1^6 + 3x_1^8]$$

```
>R3:=remseta(PS3,B3,ord);
```

$$R_3 := \{\}$$

```
>J:=Initial(B3[1],ord)*Initial(B3[2],ord)*Initial(B3[3],
ord);
```

$$J := x_1$$

```
>solveas(B3,ord,{x1});
```

这样解得 8 组解:

$$\begin{cases} x_1 = 2, \\ x_2 = 1, \\ x_3 = 2; \end{cases} \quad \begin{cases} x_1 = -2, \\ x_2 = -1, \\ x_3 = 2; \end{cases} \quad \begin{cases} x_1 = \sqrt{3}\mathrm{i}, \\ x_2 = -\dfrac{\sqrt{3}}{12}\mathrm{i}, \\ x_3 = -\dfrac{3}{2}; \end{cases} \quad \begin{cases} x_1 = -\sqrt{3}\mathrm{i}, \\ x_2 = \dfrac{\sqrt{3}}{12}\mathrm{i}, \\ x_3 = -\dfrac{3}{2}; \end{cases}$$

$$\begin{cases} x_1 = \dfrac{\sqrt{-6+6\sqrt{13}}}{3}\mathrm{i}, \\ x_2 = \dfrac{2(2+\sqrt{13})}{3\sqrt{-6+6\sqrt{13}}}\mathrm{i}, \\ x_3 = \dfrac{1-\sqrt{13}}{3}; \end{cases} \quad \begin{cases} x_1 = -\dfrac{\sqrt{-6+6\sqrt{13}}}{3}\mathrm{i}, \\ x_2 = -\dfrac{2(2+\sqrt{13})}{3\sqrt{-6+6\sqrt{13}}}\mathrm{i}, \\ x_3 = \dfrac{1-\sqrt{13}}{3}; \end{cases}$$

$$\begin{cases} x_1 = \dfrac{\sqrt{6+6\sqrt{13}}}{3}\mathrm{i}, \\ x_2 = \dfrac{2(-2+\sqrt{13})}{3\sqrt{6+6\sqrt{13}}}\mathrm{i}, \\ x_3 = \dfrac{1+\sqrt{13}}{3}; \end{cases} \quad \begin{cases} x_1 = -\dfrac{\sqrt{6+6\sqrt{13}}}{3}\mathrm{i}, \\ x_2 = -\dfrac{2(-2+\sqrt{13})}{3\sqrt{6+6\sqrt{13}}}\mathrm{i}, \\ x_3 = \dfrac{1+\sqrt{13}}{3}. \end{cases}$$

2. 解多项式方程组:

$$\begin{cases} 2x_3^2 - x_1^2 - x_2^2 = 0, \\ x_1x_3 - 2x_3 + x_1x_2 = 0, \\ x_1^2 - x_2^2 = 0. \end{cases}$$

解: 这里只列出分步法的过程, 并列出部分运算结果.

```
>read `d:/mapleuser/wsolve2`:
>P1:=2*x3^2-x1^2-x2^2;
>P2:=x1*x3-2*x3+x1*x2;
>P3:=x1^2-x2^2;
>PS1:={P1,P2,P3};
>ord:=[x3,x2,x1];
>B1:=basset(PS1,ord);
```

$$B_1 := [x_1x_3 - 2x_3 + x_1x_2, x_1^2 - x_2^2]$$

```
>R1:=remseta(PS1,B1,ord);
```

$$R_1 := \{x_1^2(x_1 - 1)\}$$

```
>PS2:={op(B1)} union R1;
>B2:=basset(PS2,ord);
```

$$B_2 := [x_1x_3 - 2x_3 + x_1x_2, x_1^2 - x_2^2, x_1^2(x_1 - 1)]$$

```
>R2:=remseta(PS2,B2,ord);
```

$$R_2 := \{\}$$

```
>J:=Initial(B2[1],ord)*Initial(B2[2],ord)*Initial(B2[3],
ord);
```

$$J := x_1 - 2$$

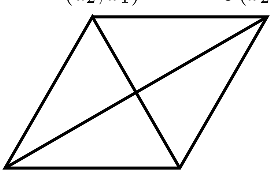
```
>solveas(B2,ord,{x1-2});
```

这样解得 4 组解:

$$\begin{cases} x_1 = 0, \\ x_2 = 0, \\ x_3 = 0, \end{cases} \quad \begin{cases} x_1 = 1, \\ x_2 = -1, \\ x_3 = -1, \end{cases} \quad \begin{cases} x_1 = 1, \\ x_2 = 1, \\ x_3 = 1. \end{cases}$$

*§ 5　几何定理的机器证明

1. 证明: 菱形的对角线互相垂直.



第 1 题

证明: 根据假设条件可以得到下列多项式方程:

$$\begin{aligned} &P_1 \overset{\text{def}}{=} u_2^2 + x_1^2 - u_1^2 = 0, && (|AD| = |AB|) \\ &P_2 \overset{\text{def}}{=} (x_2 - u_2)^2 + (x_3 - x_1)^2 - u_1^2 = 0, && (|DC| = |AB|) \\ &P_3 \overset{\text{def}}{=} (x_2 - u_1)^2 + x_3^2 - u_1^2 = 0. && (|BC| = |AB|) \end{aligned}$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2, P_3\}$.

定理结论是 $AC \perp BD$, 可以归结为多项式方程

$$G \overset{\text{def}}{=} (u_2 - u_1)x_2 + x_1x_3 = 0.$$

设变量的序为 x_1, x_2, x_3, 求得两个特征列 $\mathscr{C}_i = \{C_{i1}, C_{i2}, C_{i3}\}$ 分别为:

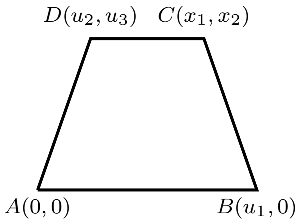
$$C_{11} = x_1^2 - u_1^2 + u_2^2,$$
$$C_{12} = x_2,$$
$$C_{13} = x_3.$$

以及

$$C_{21} = x_1^2 - u_1^2 + u_2^2,$$
$$C_{22} = -x_2 + u_1 + u_2,$$
$$C_{33} = x_1 x_3 - u_1^2 + u_2^2.$$

从 $\mathscr{C}_1$ 导出 $x_2 = x_3 = 0$, 显然是增根. 计算 $\mathrm{Rem}(G, \mathscr{C}_2) = 0$, 可知定理成立. 而非退化条件 $J_2 = x_1$, 从几何意义看, 这是不可以的.

2. 证明: 等腰梯形底角相等.



第 2 题

证明: 根据假设条件可以得到下列多项式方程:

$$P_1 \stackrel{\text{def}}{=} x_2 - u_3 = 0, \qquad (AB // CD)$$
$$P_2 \stackrel{\text{def}}{=} u_1 - x_1 - u_2 = 0. \qquad (\overrightarrow{AD} \text{ 与 } \overrightarrow{CB} \text{ 在 } AB \text{ 上的投影相等})$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2\}$.

定理结论是 $\angle BAD = \angle CBA$, 可以归结为多项式方程

$$G \stackrel{\text{def}}{=} -u_1^2 u_3 (x_1 - u_1) - u_1^2 u_2 x_2 = 0.$$

求得特征列 $\mathscr{C} = \{C_1, C_2\}$ 就是 $\mathscr{P}$ 自己, 而且 $J = 1$, 没有非退化条件. 计算 $\mathrm{Rem}(G, \mathscr{C}) = 0$, 可知定理成立.

如果把定理的第二个条件改成

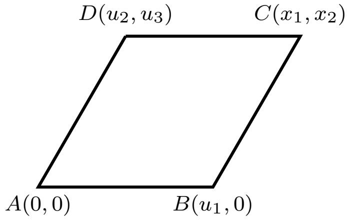
$$P_2 \stackrel{\text{def}}{=} u_2^2 + u_3^2 - (x_1 - u_1)^2 - x_2^2 = 0, \qquad (|AD| = |BC|)$$

计算后会得到两个特征列, 一个特征列同前, 另一个特征列是

$$C_1 = -x_1 + u_1 + u_2,$$

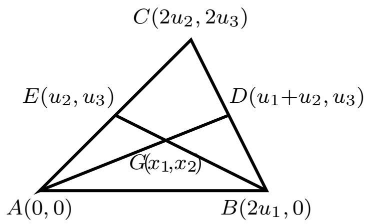
$$C_2 = x_2 - u_3.$$

G 关于这个特征列的余式等于 $u_1^2u_2u_3$, 也就是说结论不对. 从几何意义来看, $C_1 = 0$ 相当于 $u_1 = x_1 - u_2$, 即 $ABCD$ 是平行四边形 (见下图). 这是不符合题意的增根, 因此 $\angle BAD \neq \angle CBA$.



第 2 题的另一种情形

3. 证明: 三角形的两条中线的交点分顶点与对边中点成 2 : 1.



第 3 题

证明: 根据假设条件可以得到下列多项式方程:

$$P_1 \stackrel{\text{def}}{=} u_3x_1 - (u_1 + u_2)x_2 = 0, \qquad (AGD \text{ 共线})$$
$$P_2 \stackrel{\text{def}}{=} (u_2 - 2u_1)x_2 - (x_1 - 2u_1)u_3 = 0, \qquad (BGE \text{ 共线})$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2\}$.

定理结论是 $\overrightarrow{AG} = \dfrac{2}{3}\overrightarrow{AD}$, 可以归结为多项式方程

$$G_1 \stackrel{\text{def}}{=} 3x_1 - 2(u_1 + u_2) = 0,$$

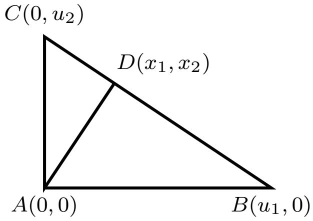
$$G_2 \stackrel{\text{def}}{=} 3x_2 - 2u_3 = 0.$$

设变量的序为 x_1, x_2, 求得特征列 $\mathscr{C} = \{C_1, C_2\}$ 为:

$$C_1 = 3x_1 - 2u_1 - 2u_2,$$
$$C_2 = -3x_2 + 2u_3.$$

而且 $J = 1$, 没有非退化条件. 显然 G_1, G_2 都能被 $\mathscr{C}$ 除尽.

4. 证明: 直角三角形斜边上的高是斜边上两线段的比例中项.



第 4 题

证明: 根据假设条件可以得到下列多项式方程:

$$P_1 \stackrel{\text{def}}{=} u_1x_1 - u_2x_2 = 0, \qquad (AD \perp BC)$$
$$P_2 \stackrel{\text{def}}{=} (x_2-u_2)u_1 + u_2x_1 = 0. \qquad (BDC\ \text{共线})$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2\}$.

定理结论是 $|AD|^2 = |CD||DB|$, 可以归结为多项式方程

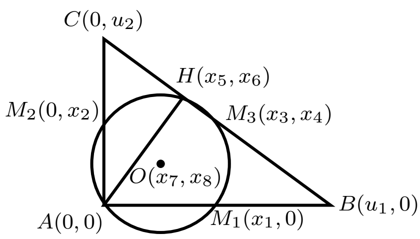
$$G \stackrel{\text{def}}{=} (x_1^2+x_2^2)^2 - (x_1^2+(x_2-u_2)^2)((u_1-x_1)^2+x_2^2) = 0.$$

设变量的序为 x_1, x_2, 求得特征列 $\mathscr{C} = \{C_1, C_2\}$ 为:

$$C_1 = (u_1^2+u_2^2)x_1 - u_1u_2^2,$$
$$C_2 = -(u_1^2-u_2^2)x_2 + u_1^2u_2.$$

计算 $\mathrm{Rem}(G, \mathscr{C}) = 0$, 可知定理成立. 而非退化条件是 u_2 以及 $J = (u_1^2+u_2^2)^2$, 从几何意义看, 这些情形都是不允许的.

5. 如图, 设 $\triangle ABC$ 中 $\angle A$ 是直角, M_1, M_2, M_3 分别是 AB, AC, BC 边的中点. $AH \perp BC$ 并且 H 是垂足. 证明 M_1, M_2, M_3, H 四点共圆.



第 5 题

证明: 根据假设条件可以得到下列多项式方程:

$$
\begin{aligned}
&P_1 \overset{\text{def}}{=} u_1 - 2x_1 = 0, && (M_1 \text{ 是 } AB \text{ 的中点})\\
&P_2 \overset{\text{def}}{=} u_2 - 2x_2 = 0, && (M_2 \text{ 是 } AC \text{ 的中点})\\
&P_3 \overset{\text{def}}{=} u_1 - 2x_3 = 0, && (M_3 \text{ 是 } BC \text{ 的中点})\\
&P_4 \overset{\text{def}}{=} u_2 - 2x_4 = 0, && (M_3 \text{ 是 } BC \text{ 的中点})\\
&P_5 \overset{\text{def}}{=} u_1x_5 - u_2x_6 = 0, && (AH \perp BC)\\
&P_6 \overset{\text{def}}{=} u_1(x_6 - u_2) + u_2x_5 = 0. && (BHC \text{ 共线})\\
&P_7 \overset{\text{def}}{=} (x_1 - x_7)^2 - x_7^2 = 0, && (|OM_1| = |OA|)\\
&P_8 \overset{\text{def}}{=} (x_2 - x_8)^2 - x_8^2 = 0. && (|OM_2| = |OA|)
\end{aligned}
$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2, \cdots, P_8\}$.

定理结论是 $|OH| = |OM_3| = |OA|$, 可以归结为多项式方程

$$G_1 \overset{\text{def}}{=} (x_5 - x_7)^2 + (x_6 - x_8)^2 - x_7^2 - x_8^2 = 0,$$

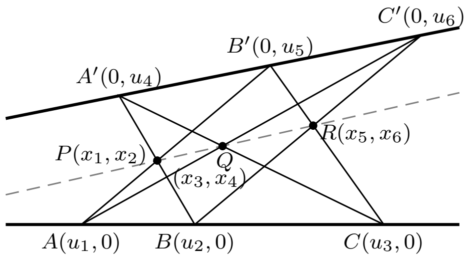
$$G_2 \overset{\text{def}}{=} (x_3 - x_7)^2 + (x_4 - x_8)^2 - x_7^2 - x_8^2 = 0.$$

设变量的序为 $x_1, x_2, \cdots, x_8$, 求得特征列 $\mathscr{C} = \{C_1, C_2, \cdots, C_8\}$ 为:

$$
\begin{aligned}
C_1 &= -2x_1 + u_1,\\
C_2 &= -2x_2 + u_2,\\
C_3 &= -2x_3 + u_1,\\
C_4 &= -2x_4 + u_2,\\
C_5 &= (u_1^2 + u_2^2)x_5 - u_1u_2^2,\\
C_6 &= -(u_1^2 + u_2^2)x_6 + u_1^2u_2,\\
C_7 &= -4x_7 + u_1,\\
C_8 &= -4x_8 + u_2.
\end{aligned}
$$

计算 $\mathrm{Rem}(G_1, \mathscr{C}) = 0$, $\mathrm{Rem}(G_2, \mathscr{C}) = 0$, 可知定理成立. 而非退化条件是 u_2 以及 $J = (u_1^2 + u_2^2)^2$, 从几何意义看, 这些情形都是不允许的.

6. 如图, A, B, C 三点在一条直线上, A', B', C' 三点在另一条直线上. P, Q, R 是它们连线的交点. 证明: P, Q, R 三点共线.



第 6 题

证明: 因为这是个仿射问题, 因此可建立仿射坐标系如上图所示.

根据假设条件可以得到下列多项式方程:

$$
\begin{aligned}
&P_1 \stackrel{\text{def}}{=} x_1(-u_4)-u_2(x_2-u_4)=0, &&(A'PB\text{ 共线})\\
&P_2 \stackrel{\text{def}}{=} x_1(-u_5)-u_1(x_2-u_5)=0, &&(B'PA\text{ 共线})\\
&P_3 \stackrel{\text{def}}{=} x_3(-u_4)-u_3(x_4-u_4)=0, &&(A'QC\text{ 共线})\\
&P_4 \stackrel{\text{def}}{=} x_3(-u_6)-u_1(x_4-u_6)=0, &&(C'QA\text{ 共线})\\
&P_5 \stackrel{\text{def}}{=} x_5(-u_5)-u_3(x_6-u_5)=0, &&(B'RC\text{ 共线})\\
&P_6 \stackrel{\text{def}}{=} x_5(-u_6)-u_2(x_6-u_6)=0. &&(C'RB\text{ 共线})
\end{aligned}
$$

这样定理假设可以归结成一个多项式组 $\mathscr{P}=\{P_1,P_2,\cdots,P_6\}$.

定理结论是 PQR 共线, 可以归结为多项式方程

$$G \stackrel{\text{def}}{=} (x_3-x_1)(x_6-x_2)-(x_5-x_1)(x_4-x_2)=0,$$

设变量的序为 $x_1,x_2,\cdots,x_6$, 求得特征列 $\mathscr{C}=\{C_1,C_2,\cdots,C_6\}$ 为:

$$
\begin{aligned}
C_1&=(u_1u_4-u_2u_5)x_1-u_1u_2(u_4-u_5),\\
C_2&=(u_1u_4-u_2u_5)x_2-(u_1-u_2)u_4u_5,\\
C_3&=(u_1u_4-u_3u_6)x_3-u_1u_3(u_4-u_6),\\
C_4&=(u_1u_4-u_3u_6)x_4-(u_1-u_3)u_4u_6,\\
C_5&=-(u_2u_5-u_3u_6)x_5+u_2u_3(u_5-u_6),\\
C_6&=-(u_2u_5-u_3u_6)x_6+(u_2-u_3)u_5u_6.
\end{aligned}
$$

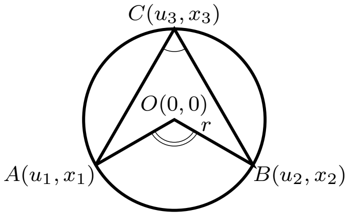
计算 $\operatorname{Rem}(G,\mathscr{C})=0$, 可知定理成立. 而非退化条件是 u_2,u_3 以及 $J=(u_1u_4-u_2u_5)^2(u_1u_4-u_3u_6)^2(u_2u_5-u_3u_6)^2$.

对非退化条件的几何意义作分析:

(1) 若 $u_2=0$ 或 $u_3=0$, 它的几何意义是 B 或 C 与 $A'B'C'$ 共线, 从而 P,Q,R 不确定, 问题无意义.

(2) $u_1u_4 - u_2u_5 = 0 \implies A'B//B'A$, $u_1u_4 - u_3u_6 = 0 \implies A'C//C'A$, $u_2u_5 - u_3u_6 = 0 \implies B'C//C'B$, 任何一种情形出现都会使 P, Q, R 中的一个点无法确定, 问题无意义.

7. 证明: 圆心角等于相应圆周角的两倍.



第 7 题

证明: 根据假设条件可以得到下列多项式方程:

$$P_1 \stackrel{\text{def}}{=} u_1^2 + x_1^2 - r^2 = 0, \qquad (|OA| = r)$$
$$P_2 \stackrel{\text{def}}{=} u_2^2 + x_2^2 - r^2 = 0, \qquad (|OB| = r)$$
$$P_3 \stackrel{\text{def}}{=} u_3^2 + x_3^2 - r^2 = 0, \qquad (|OC| = r)$$

这样定理假设可以归结成一个多项式组 $\mathscr{P} = \{P_1, P_2, P_3\}$.

定理结论是 $\angle AOB = 2\angle ACB$, 即

$$\tan \angle AOB = \tan(2\angle ACB) = \frac{2\tan \angle ACB}{1 - \tan^2 \angle ACB}. \qquad (*)$$

由于

$$\angle AOB = \frac{k_{OB} - k_{OA}}{1 + k_{OB}k_{OA}} = \frac{u_1x_2 - u_2x_1}{u_1u_2 + x_1x_2},$$

$$\angle ACB = \frac{k_{CB} - k_{CA}}{1 + k_{CB}k_{CA}} = \frac{(u_1 - u_3)(x_2 - x_3) - (u_2 - u_3)(x_1 - x_3)}{(u_1 - u_3)(u_2 - u_3) + (x_1 - x_3)(x_2 - x_3)} = \frac{\alpha}{\beta}.$$

代入 $(*)$ 式得

$$\frac{u_1x_2 - u_2x_1}{u_1u_2 + x_1x_2} = \frac{2\frac{\alpha}{\beta}}{1 - \frac{\alpha^2}{\beta^2}} = \frac{2\alpha\beta}{\beta^2 - \alpha^2}.$$

因此命题的结论可以归结为多项式方程

$$\begin{aligned} G \stackrel{\text{def}}{=} & (u_1x_2 - u_2x_1)(((u_1 - u_3)(u_2 - u_3) + (x_1 - x_3)(x_2 - x_3))^2 \\ & - ((u_1 - u_3)(x_2 - x_3) - (u_2 - u_3)(x_1 - x_3))^2) - 2(u_1u_2 + x_1x_2) \\ & \times ((u_1 - u_3)(x_2 - x_3) - (u_2 - u_3)(x_1 - x_3))((u_1 - u_3)(u_2 - u_3) \\ & + (x_1 - x_3)(x_2 - x_3)) = 0. \end{aligned}$$

设变量的序为 x_1, x_2, x_3, 求得特征列 $\mathscr{C}$ 就是原来的多项式组 $\mathscr{P}$. 计算 $\mathrm{Rem}(G, \mathscr{C}) = 0$, 可知定理成立. 而且没有非退化条件.