Determination of Fiber Orientation in MRI Diffusion Tensor Imaging Based on Higher-Order Tensor Decomposition

Leslie (Lei) Ying, Yi Ming Zou, David P. Klemer and Jiun-Jie Wang

Abstract—High Angular Resolution Diffusion Imaging (HARDI) techniques have been used for resolving multiple fiber directions within a voxel. Using HARDI, a high-order tensor can be obtained through generalized diffusion tensor imaging (GDTI). In this paper, based on the decomposition of the high-order diffusion tensors, a mathematical technique is presented which permits accurate resolution of multiple, randomly-oriented fiber tracts within tissue. A sequence of pseudo-eigenvalues and pseudo-eigenvectors are derived from the diffusion tensor through successive application of a best least-square rank-1 tensor approximation. These pseudo-eigenvalues and pseudo-eigenvectors are used to identify the major fiber directions within an individual image voxel. Results of a numerical simulation are presented to demonstrate the technique.

I. INTRODUCTION

DIFFUSION tensor imaging (DTI) techniques have been applied to numerous applications in biomedical research – from white-matter tractography to post-infarction myocardial remodeling – as well as clinical investigations, such as use in pre-surgical planning for intracranial tumor resection as well as image localization of neuropathological lesions involving axonal tracts.

In cases where a single fiber direction predominates, such as the optic radiations from the lateral geniculate nucleus to the visual cortex, DTI techniques perform well, providing useful structural detail of axonal course [1]. Additional work is needed, however, in accurately resolving fiber directions within regions of tissue containing numerous fibers or fiber tracts oriented in multiple non-parallel directions. High Angular Resolution Diffusion Imaging (HARDI) techniques have been proposed to address this problem [2].

Using HARDI, Özarslon, et al. [3], have developed generalized techniques for imaging of diffusion profiles within tissue; these generalized Diffusion Tensor Imaging (GDTI) methods represent diffusivity through the use of supersymmetric tensors of order higher than 2, leading to the formulation of a generalized Stejskal-Tanner equation. (N.B.: In this paper, we use the term “order” rather than the term “rank”, since the term “rank” has its fixed meaning). In this approach, the components of a supersymmetric high order tensor can be determined by fitting the HARDI data.

It has been hypothesized that in DTI, the principal eigenvector of a second-order tensor (i.e., a matrix) can provide information about local fiber orientation in tissue, and this has been verified in the case where a fiber bundle with approximately the same orientation passing through a voxel [5]. Though it is easy to obtain the eigenvalues and the eigenvectors of a second-order tensor, a problem with the second-order tensor is that the eigenvectors corresponding to different eigenvalues are always orthogonal, so the other two eigenvectors can not be reliably used to predict different directions at the fiber crossing. On the other hand, to abstract information from a higher-order tensor is more complicated: unlike the orthogonal diagonalization of a second-order tensor, the diagonal and the orthogonal decompositions of a higher-order tensor are in general not compatible: the higher-order tensor can be decomposed with either a diagonal core tensor and non-orthogonal vectors (a PARAFAC decomposition), or a non-diagonal core tensor with orthogonal vectors (Tucker or HOSVD decomposition), but not both. In addition, the computation of such decompositions is still a topic of investigation and current techniques [6] do not yield unique solutions in general, therefore are unsuitable for determination of tissue fiber direction.

Existing GDTI allows for the visualization of fiber heterogeneity in tissue based on a three-dimensional mapping of (water) diffusivity [3]. In the case where fiber orientations in tissue are not necessarily orthogonal, we propose the method described here, in which a greedy PARAFAC decomposition is employed. Using this technique, multiple fiber directions may be recovered from a set of HARDI data. In the following sections, results of numerical simulations are presented which demonstrate the ability of this method to simultaneously determine fiber directions accurately.
II. Theory

A. Overview of GDTI

Consider a high-order Cartesian tensor $D$, represented as:

$$D \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}.$$  \hfill (1)

The generalized DTI technique extends the Bloch-Torrey equation to include a phenomenological diffusion term with a high order Cartesian tensor (in this case, all indices $I_i = 3$):

$$\frac{\partial M_+}{\partial t} = -i\omega_0 M_+ - i\gamma \cdot G M_+ - M_+ / T_2 +$$

$$\sum_{i=1}^{3} \sum_{i=1}^{3} \cdots \sum_{i=1}^{3} D_{i12\cdots iN} g_{i1} g_{i2} \cdots g_{iN} \nabla^2 M_+$$  \hfill (2)

where $M_+ = M_x + iM_y$ is the complex representation of transverse magnetization, $r$ is the position vector, $G$ is the linear magnetic field gradient, $\gamma$ is the gyromagnetic ratio, $\omega_0$ is the Larmor frequency, $T_2$ is the spin-spin relaxation time, $D_{i12\cdots iN}$ are the components of the tensor $D$, and $g_{i1} \cdots g_{iN}$ are components of the gradient vector [3].

The generalized Stejskal-Tanner formula is given by

$$\ln S = \ln S_0 - b \times \sum_{i=1}^{3} \sum_{i=1}^{3} \cdots \sum_{i=1}^{3} D_{i12\cdots iN} g_{i1} g_{i2} \cdots g_{iN}.$$  \hfill (3)

Based on this formula, the high-order tensor $D$ can be obtained from the diffusion weighted MR signal by means of a simple multilinear regression. The diffusivity profile can be obtained from:

$$D(g) = \sum_{i=1}^{3} \sum_{i=1}^{3} \cdots \sum_{i=1}^{3} D_{i12\cdots iN} g_{i1} g_{i2} \cdots g_{iN}.$$  \hfill (4)

A general tensor of this kind has $3^N$ terms. However, the diffusion tensor is known to be supersymmetric; i.e., the entries are invariant under any permutation of their indices. This property results in a significant reduction in the number of distinct components, given by $(N+1)(N+2)/2$ for a tensor of order $N$.

B. The Best Rank-1 Tensor Approximation

The rank $R$ of a tensor of order $N$ is defined as the minimal number of terms in a finite decomposition of $D$ of the form

$$D = \sum_{r=1}^{N} \mathbf{v}_r^{(1)} \otimes \mathbf{v}_r^{(2)} \otimes \cdots \otimes \mathbf{v}_r^{(N)},$$  \hfill (5)

where $\mathbf{v}_r^{(i)}$ are column vectors of the right dimensions and $\otimes$ denotes the tensor product. As pointed out in reference [8], the decomposition of a supersymmetric tensor $D$ results in column vectors $\mathbf{v}_r^{(i)}$ to be equal to each other for different indices $i$. In this context, the best rank-1 approximation of a tensor $D$ is defined by a scalar $\lambda > 0$, and a corresponding vector $\mathbf{v}$ which has unit norm. The value of $\lambda$ and $\mathbf{v}$ are obtained by minimizing the norm:

$$\|D - \lambda_0 \mathbf{v}_0 \otimes \mathbf{v}_0 \otimes \cdots \otimes \mathbf{v}_0\|_F^2 =$$

$$\sum_{i=1}^{N} (D_{i12\cdots iN} - \lambda v_0^{(1)} v_0^{(2)} \cdots v_0^{(N)})^2,$$  \hfill (6)

where $\|\cdot\|_F$ denotes the Frobenius norm [7]. The scalar $\lambda$ is termed a pseudo-eigenvalue, and the unit-norm vector $\mathbf{v}$ the corresponding pseudo-eigenvector of tensor $D$. We use the so-called symmetric higher-order power method (S-HOPM, [8]) to find the best rank-1 tensor approximation as follows:

\begin{itemize}
  \item \textbf{Initialization: choose} $\mathbf{v}^{(0)}$, a unit-norm vector.
  \item \textbf{Iteration:}
    \begin{itemize}
      \item for $k = 1, 2, \ldots$
      \end{itemize}
    \end{itemize}

$$\mathbf{v}^{(k)} = \frac{\mathbf{D}(\mathbf{v} \otimes \cdots \otimes \mathbf{v})}{\|\mathbf{v}\|}, \quad \lambda = \frac{\mathbf{D}(\mathbf{v} \otimes \cdots \otimes \mathbf{v})}{\|\mathbf{v}\|}$$  \hfill (7)

where $\mathbf{D}(\mathbf{v} \otimes \cdots \otimes \mathbf{v})$ is defined by $\sum_{i=1}^{N} D_{i12\cdots iN} v_1 \cdots v_N$, and

$$\mathbf{D}(\mathbf{v} \otimes \cdots \otimes \mathbf{v}) = \sum_{i=1}^{N} D_{i12\cdots iN} v_1 \cdots v_N.$$  \hfill (8)

C. Greedy PARAFAC Decomposition

The proposed technique is based on the decomposition of an even-order supersymmetric tensor $D$ obtained from generalized DTI, using a greedy PARAFAC method to accomplish the decomposition through successive application of the best least-square rank-1 tensor approximation.

The minimization problem is then solved iteratively. After determining the appropriate values $\lambda_0$ and $\mathbf{v}_0$ for the first iteration, the residual $D_1$ is computed as

$$D_1 = D - \lambda_0 \mathbf{v}_0 \otimes \mathbf{v}_0 \otimes \cdots \otimes \mathbf{v}_0.$$  \hfill (9)
Then the minimization process is repeated by applying the same best rank-1 approximation to the residual $D_1$ to obtain a second set of pseudo-eigenvalue and pseudo-eigenvector, $\lambda_1$ and $\mathbf{v}_1$. We continue to iteratively repeat this procedure until the magnitude of the residual tensor $D_n$ is sufficiently small so as to satisfy a given termination criterion, completing the decomposition process.

Finally, the directions of the pseudo-eigenvectors corresponding to the $r$ pseudo-eigenvalues with the greatest magnitude reveal the directions of $r$ fibers. When applied to order-2 tensors, it is worth noting that this method determines the eigenvalues and eigenvectors exactly. Also, this decomposition does not assume orthogonality between pseudo-eigenvectors. This can be justified by investigating the inverse Fourier transform of the diffusion profile. Under the assumption of linearly independence of the pseudo-eigenvectors, we can apply coordinate transformations to the following integral which relates the MR signal attenuation and the average displacement probability (c.f. equation (1) in [4]):

$$P(\mathbf{r}) = S_0 \int e^{-bD(g)} e^{i\mathbf{g} \cdot \mathbf{r}} d\mathbf{g}.$$  

After suitable coordinate transformations, the above inverse Fourier transform is reduced to a product of integrals:

$$P(\mathbf{r}) \propto \prod_i \int e^{-x_i N + i a_i(\mathbf{r}) x_i} dx_i,$$

where $a_i(\mathbf{r})$ denotes a linear coordinate transformation. For small values of $N$, say $N = 4$ or 6, this last integral can be evaluated exactly. For example, when $N = 4$, the integration results in the probability $P(\mathbf{r})$ being expressed in terms of hypergeometric functions; and from this formula, it is straightforward to check the graph to confirm the directions of the pseudo-eigenvectors indicates the maximal directions of the probability profile $P(\mathbf{r})$. Details of these mathematical arguments will appear in a full paper.

III. METHOD AND RESULTS

A mathematical solution of simulated non-parallel fiber crossings was accomplished in a MATLAB environment (MathWorks, Natick, MA) in order to validate the proposed technique for determination of fiber direction. An exact form [9] of the diffusion-weighted magnetic resonance signal generated by particle diffusion within a closed cylinder was employed to simulate HARDI data, and a tensor of order 6 computed using the GDTI technique.

The numerical simulation was based on a voxel which contained three coplanar crossing fibers constrained to lie in the horizontal (xy) plane and assuming angles of $\pi/12$, $5\pi/12$, and $3\pi/4$ radians with respect to the positive x-axis, as shown in Figure 1(a). The vector representations of the three fiber directions are given by $[0.96 0.26 0]^T$, $[0.26 0.96 0]^T$, and $[0.71 -0.71 0]^T$. Given the known orientation of the crossing fibers, an apparent diffusion profile can be computed for the three crossing fibers – shown in Figure 1(b) – and from this diffusivity mapping, the order-6 tensor can then be computed as described before.

![Fig. 1. Illustration of fiber orientations used in numerical simulations: (a) true orientations of three cylinders; (b) associated diffusivity profile.](image)

The proposed tensor decomposition technique can then be applied to the computed tensor. This results in a number of pseudo-eigenvalues and associated pseudo-eigenvectors. The five pseudo-eigenvalues with the greatest magnitudes, along with their corresponding pseudo-eigenvectors, are given by (in order of descending magnitude):

$$\begin{align*}
\lambda_0 &= 2.97, \quad \mathbf{v}_0 = [-0.71 0.71 0]^T, \\
\lambda_1 &= 2.93, \quad \mathbf{v}_1 = [0.27 0.96 0]^T, \\
\lambda_2 &= 2.87, \quad \mathbf{v}_2 = [0.96 0.25 0]^T, \\
\lambda_3 &= 1.52, \quad \mathbf{v}_3 = [-0.70 -0.71 0]^T, \\
\lambda_4 &= 1.47, \quad \mathbf{v}_4 = [-0.96 0.27 0]^T.
\end{align*}$$

The first three pseudo-eigenvalues have values which are roughly similar in magnitude, having a mean value of 2.92 and standard deviation of 0.050; the last two pseudo-eigenvalues are significantly less in magnitude. Choosing only those pseudo-eigenvectors corresponding to the highest-valued pseudo-eigenvalues yields the three original known fiber directions. The maximum angular error in the fiber directions reconstructed using this method is 0.01 radians.

IV. DISCUSSION AND CONCLUSIONS

Through interative decomposition of a higher order diffusion tensor, the proposed method can accurately reveal the orientation of major fibers within an image voxel. Such a technique may be of great utility in regions of tissue in which there does not exist a single, predominant orientation of fiber direction, but rather multiple major fiber directions – for example, the region of the optic chiasm above the pituitary gland, in which there may be two or more major
tract orientations. Other examples are possibly the thalamus and basal ganglia, in which ascending and descending tracts are integrated in the control of movement, along with recirculating feedback fibers which serve to regulate motor activity. We also submit that this technique may be useful in resolving reentrant or aberrant tracts in the cardiac conduction system that may play a role in noninvasive clinical assessment of cardiac arrhythmias.

Although the higher-magnitude pseudo-eigenvalues which are derived in this technique serve to correctly identify the major fiber orientations, additional research investigation is needed to clearly define appropriate cutoff criteria to correctly assess the number of crossing fibers actually present within a voxel.

In addition, the proposed method is based on a higher-rank tensor obtained using the GDTI method, and it is well-known that GDTI assumes a signal attenuation along an arbitrary direction which is monoexponential, i.e., Gaussian diffusion. Although this may not be a completely accurate model in some situations – for example, in the case where restrictions on diffusion are present – previous studies have shown that this has an insignificant effect in the computed results. Nevertheless, non-Gaussian diffusion can also be described by a set of higher-order tensors [9], for which the tensor decomposition techniques described herein will also be applicable, with some modification. Investigation of the case where non-Gaussian diffusion is present will serve as the subject of future research work.

REFERENCES


