Variable Exponent Functionals in Image Restoration

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Introduction

Noise Model

\[ f = u + n \]

\( f \)–observed image, \( n \)–additive noise, \( u \)–true image
BV function

- Image as function belongs to the BV space. Assume \( u \in L^1(\Omega) \), we call \( u \in BV \) if \( u \) satisfies

\[
\int_{\Omega} |Du| = \sup_{|\phi| \leq 1} \left\{ \int_{\Omega} u \text{div}(\phi) \big| \phi \in C^1_0(\Omega, \mathbb{R}^2) \right\} < \infty
\]

\( \int_{\Omega} |Du| \) is the total variation of \( u \).
 BV function

- Image as function belongs to the BV space. Assume $u \in L^1(\Omega)$, we call $u \in BV$ if $u$ satisfies
  $$\int_{\Omega} |Du| = \sup_{|\phi| \leq 1} \left\{ \int_{\Omega} u \text{div}(\phi) \right\} \phi \in C^1_0(\Omega, \mathbb{R}^2) < \infty$$
  where $|Du|$ is the total variation of $u$.

- Co-area formula:
  $$\int_{\Omega} |Du| = \int_{-\infty}^{+\infty} \text{Per}(\{x : u(x) > \mu\}) \, d\mu$$
**BV function**

- Image as function belongs to the BV space. Assume $u \in L^1(\Omega)$, we call $u \in BV$ if $u$ satisfies
  \[
  \int_{\Omega} |Du| = \sup_{|\phi| \leq 1} \left\{ \int_{\Omega} u \text{div}(\phi) \mid \phi \in C^1_0(\Omega, \mathbb{R}^2) \right\} < \infty
  \]
  $\int_{\Omega} |Du|$ is the total variation of $u$.

- Co-area formula:
  \[
  \int_{\Omega} |Du| = \int_{-\infty}^{+\infty} \text{Per}(\{ x : u(x) > \mu \}) d\mu
  \]

- If $u \in C^1$, then $\int_{\Omega} |Du| = \int_{\Omega} |\nabla u| dx$. 

Constant exponent model

Rudin, Osher and Fatemi (1992)

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

(1.1)
Constant exponent model

Rudin, Osher and Fatemi (1992)

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

(1.1)

- \( p = 1 \Rightarrow \) ROF model (anisotropic diffusion model)
Constant exponent model

Rudin, Osher and Fatemi (1992)

$$\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}$$

(1.1)

- $p = 1 \Rightarrow$ ROF model (anisotropic diffusion model)
- preserve edges
Constant exponent model

Rudin, Osher and Fatemi (1992)

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

\( p = 1 \Rightarrow \) ROF model (anisotropic diffusion model)

- preserve edges
- staircase effect
Constant exponent model

Isotropic diffusion model

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\] (1.1)
Constant exponent model

Isotropic diffusion model

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} \quad (1.1)
\]

- \( p = 2 \Rightarrow \) isotropic diffusion model
**Constant exponent model**

**Isotropic diffusion model**

\[
\min \left\{ E(u) = \int_\Omega |\nabla u|^p \, dx + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, dx \right\} \tag{1.1}
\]

- \( p = 2 \) \( \Rightarrow \) isotropic diffusion model
- no *staircase* effect
Constant exponent model

Isotropic diffusion model

\[
\min \left\{ \int_{\Omega} |\nabla u|^p \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} 
\]

\[ p = 2 \Rightarrow \text{isotropic diffusion model} \]

- no \textit{staircase} effect
- blurred or dislocated the edges
Variable exponent model

Blomgren, Chan, Mulet and Wang (1997)

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^{p(|\nabla u|)} \, dx \right\}
\]

(1.2)

where \( \lim_{s \to 0} p(s) = 2 \), \( \lim_{s \to \infty} p(s) = 1 \), and \( p \) is monotonically decreasing.
Variable exponent model

Blomgren, Chan, Mulet and Wang (1997)

\[
\min \left\{ E(u) = \int_\Omega |\nabla u|^{p(|\nabla u|)} \, dx \right\}
\]  \hspace{1cm} (1.2)

where \( \lim_{s \to 0} p(s) = 2, \lim_{s \to \infty} p(s) = 1 \), and \( p \) is monotonically decreasing.

- Reduced *staircase* effect, preserve edges
Variable exponent model

Blomgren, Chan, Mulet and Wang (1997)

\[
\min \left\{ E(u) = \int_{\Omega} |\nabla u|^{p(|\nabla u|)} \,dx \right\}
\]

(1.2)

where \( \lim_{s \to 0} p(s) = 2 \), \( \lim_{s \to \infty} p(s) = 1 \), and \( p \) is monotonically decreasing.

- Reduced *staircase* effect, preserve edges
- Difficult in theoretical analysis
Variable exponent model


\[
\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left\{ E(u) = \int_{\Omega} \varphi(x, Du) + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dx \right\} \tag{1.3}
\]

where

\[
\varphi(x, r) := \begin{cases} 
\frac{1}{q(x)} |r|^{q(x)}, & |r| \leq \beta \\
|r| - \beta q(x) - \beta q(x), & |r| > \beta
\end{cases}
\]

and \( q(x) = 1 + \frac{1}{1 + k |\nabla G_\sigma * f(x)|} \) \( G_\sigma(x) \) is the Gauss kernel.
Variable exponent model


\[
\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left\{ E(u) = \int_{\Omega} \phi(x, Du) + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} \quad (1.3)
\]

where

\[
\phi(x, r) := \begin{cases} 
\frac{1}{q(x)} |r|^{q(x)}, & |r| \leq \beta \\
|r| - \frac{\beta q(x) - \beta q(x)}{q(x)}, & |r| > \beta 
\end{cases}
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- Reduced staircase effect, preserve edges
Variable exponent model


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\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left\{ E(u) = \int_{\Omega} \varphi(x, Du) + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} \quad (1.3)
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where

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\varphi(x, r) := \begin{cases} 
\frac{1}{q(x)} |r|^{q(x)}, & |r| \leq \beta \\
|r| - \frac{\beta q(x) - q(x)}{q(x)}, & |r| > \beta 
\end{cases}
\]

and \( q(x) = 1 + \frac{1}{1 + k|\nabla G_{\sigma} * f(x)|} \) \( G_{\sigma}(x) \) is the Gauss kernel.

- Reduced \textit{staircase} effect, preserve edges
- Energy minimizing problem and the associated heat flow are studied.
Variable exponent model

The proposed model

\[
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

(1.4)

where \( p(x) = 1 + g(x), \ g(x) = \frac{1}{1 + k|\nabla G_\sigma * f(x)|}. \)
Variable exponent model

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\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

where \( p(x) = 1 + g(x) \), \( g(x) = \frac{1}{1 + k |\nabla G_\sigma * f(x)|} \).

- In the region with edges, \( g \rightarrow 0 \), (1.4) approximates ROF model, so edges will be preserved.
Variable exponent model

The proposed model

\[
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
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(1.4)

where \( p(x) = 1 + g(x) \), \( g(x) = \frac{1}{1 + k |\nabla G_\sigma * f(x)|} \).

- In the region with edges, \( g \to 0 \), (1.4) approximates ROF model, so edges will be preserved.
- In relative smooth regions \( g \to 1 \), (1.4) approximates isotropic smoothing, so they will be processed into piecewise smooth regions;
Variable exponent model

The proposed model

\[
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ \mathcal{E}_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]  

(1.4)

where \( p(x) = 1 + g(x), \ g(x) = \frac{1}{1 + k |\nabla G_\sigma * f(x)|} \).

- In the region with edges, \( g \to 0 \), (1.4) approximates ROF model, so edges will be preserved.
- In relative smooth regions \( g \to 1 \), (1.4) approximates isotropic smoothing, so they will be processed into piecewise smooth regions;
- In other regions, the diffusion is properly adjusted by the function \( p(x) \).
Variable exponent model

The proposed model

\[
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\}
\]

where \( p(x) = 1 + g(x), \ g(x) = \frac{1}{1+k|\nabla G_\sigma*f(x)|} \).

- In the region with edges, \( g \to 0 \), (1.4) approximates ROF model, so edges will be preserved.
- In relative smooth regions \( g \to 1 \), (1.4) approximates isotropic smoothing, so they will be processed into piecewise smooth regions;
- In other regions, the diffusion is properly adjusted by the function \( p(x) \).
- Energy minimizing problem and the associated heat flow are studied.
Variable exponent space $L^{p(x)}$, $W^{1,p(x)}$

Let $p(x) : \Omega \to [1, +\infty)$ be a measurable function, called variable exponent on $\Omega$. By $\mathcal{P}(\Omega)$ we denote the family of all measurable functions on $\Omega$. We write $p^- := \text{ess inf}_\Omega p(x)$, $p^+ := \text{ess sup}_\Omega p(x)$. We define a functional

$$Q_{p(x)}(u) = \int_\Omega |u|^{p(x)} \, dx$$

and a norm by formula

$$\|u\|_{p(x)} = \|u\|_{L^{p(x)}(\Omega)} := \inf\{\lambda > 0 : Q_{p(x)}(u/\lambda) \leq 1\}$$

Then the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ are defined as
Variable exponent space $L^{p(x)}$, $W^{1,p(x)}$

\[ L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid \|u\|_{p(x)} < \infty \right\} \]

\[ W^{1,p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \in L^{p(x)}(\Omega), \nabla u \in L^{p(x)}(\Omega) \right\} \]

The norm \( \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \) makes $W^{1,p(x)}(\Omega)$ a Banach space. $W^{1,p(x)}_0(\Omega)$ denotes the closure of $C_0^{\infty}(\Omega)$ under the norm $\| \cdot \|_{1,p(x)}$. 
Variable exponent space
Properties of variable exponent space

- **Lemma 2.1** Let $p(x), q(x) \in \mathcal{P}(\Omega)$, and for a.e. $x \in \Omega$ we have $p(x) \leq q(x)$. Then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, $W^{1,q(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$. The norm of the embedding operator does not exceed $1 + |\Omega|$, where $|\Omega|$ denotes the measure of $\Omega$. 
Properties of variable exponent space

**Lemma 2.1** Let $p(x), q(x) \in \mathcal{P}(\Omega)$, and for a.e. $x \in \Omega$ we have $p(x) \leq q(x)$. Then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, $W^{1,q(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$. The norm of the embedding operator does not exceed $1 + |\Omega|$, where $|\Omega|$ denotes the measure of $\Omega$.

**Lemma 2.2** Let $p(x) \in \mathcal{P}(\Omega), 1 < p^− \leq p^+ < \infty$. Then $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are all reflexive Banach space.
Properties of variable exponent space

**Lemma 2.1** Let $p(x), q(x) \in \mathcal{P}(\Omega)$, and for a.e. $x \in \Omega$ we have $p(x) \leq q(x)$. Then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, $W^{1,q(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$. The norm of the embedding operator does not exceed $1 + |\Omega|$, where $|\Omega|$ denotes the measure of $\Omega$.

**Lemma 2.2** Let $p(x) \in \mathcal{P}(\Omega)$, $1 < p^- \leq p^+ < \infty$. Then $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are all reflexive Banach space.

**Lemma 2.3** Let $F(\nabla u, u, x) = \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\lambda}{2} (u - f)^2$, $p(x) = 1 + g(x)$ as in model (1.4). Then for each $z, x$, $F(\xi, z, x)$ is convex in $\xi$. 

Properties of variable exponent space

**Lemma 2.4** Let $F(\xi, z, x)$ be bounded from below, and the map $\xi \mapsto F(\xi, z, x)$ is convex in each $z \in \mathbb{R}, x \in \Omega$. Then the energy functional $I(u) := \int_{\Omega} F(\nabla u, u, x)dx$ is lower semi-continuous in $W^{1, p(x)}(\Omega)$. 
Properties of variable exponent space

- **Lemma 2.4** Let $F(\xi, z, x)$ be bounded from below, and the map $\xi \mapsto F(\xi, z, x)$ is convex in each $z \in \mathbb{R}, x \in \Omega$. Then the energy functional $I(u) := \int_{\Omega} F(\nabla u, u, x)dx$ is lower semi-continuous in $W^{1,p(x)}(\Omega)$.

- **Lemma 2.5** Let the dimension of $\Omega$ be $n = 2$, $1 \leq p^- \leq p(x) \leq p^+ \leq 2$. Then the embedding $W^{1,p(x)} \hookrightarrow L^{p(x)}$ is compact.
Properties of variable exponent space

Lemma 2.4  Let $F(\xi, z, x)$ be bounded from below, and the map $\xi \mapsto F(\xi, z, x)$ is convex in each $z \in \mathbb{R}, x \in \Omega$. Then the energy functional $I(u) := \int_\Omega F(\nabla u, u, x) dx$ is lower semi-continuous in $W^{1,p(x)}(\Omega)$.

Lemma 2.5  Let the dimension of $\Omega$ be $n = 2$, $1 \leq p^- \leq p(x) \leq p^+ \leq 2$. Then the embedding $W^{1,p(x)} \hookrightarrow L^{p(x)}$ is compact.

Proof  $n = 2$,

$$(p^-)^* = \frac{np^-}{n-p^-} = \frac{n}{n/p^- - 1} \geq \frac{n}{n-1} = 2 \geq p^+.$$

$$W^{1,p(x)}(\Omega) \hookrightarrow W^{1,p^-}(\Omega) \hookrightarrow L^{p^+}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

$$W^{1,p^-}(\Omega) \hookrightarrow L^{p^+}(\Omega) \text{ compact.}$$
The existence and uniqueness of minimizer I

Theorem 2.1  Let $\Omega \subset \mathbb{R}^2$ be bounded open set with Lipschitz boundary, $f \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Then the minimization problem

$$
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} \quad (2.1)
$$

has unique minimizer $u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$.
The minimization problem

The existence and uniqueness of minimizer

Proof Let \( \mu = \inf_{v \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)} E_{p(x)}(v) \). Since \( f \in W^{1,p(x)}(\Omega) \cap L^2(\Omega) \), \( \mu \) is finite. Let \( \{u_k\}_{k=1}^{\infty}, u_k \in W^{1,p(x)}(\Omega) \cap L^2(\Omega) \) be the minimizing sequence such that \( E_{p(x)}(u_k) \to \mu \). Then

\[
\int_{\Omega} \frac{1}{p(x)} |\nabla u_k|^{p(x)} \, dx \leq C \quad \text{and} \quad \int_{\Omega} (u_k - f)^2 \, dx \leq C
\]

Hence \( \int_{\Omega} (u_k)^2 \, dx \leq C \). By Lemma 2.1, \( L^2(\Omega) \subset L^{p(x)}(\Omega) \). So we have \( \int_{\Omega} |u|^{p(x)} \, dx \leq C \). Together with the inequality

\[
\int_{\Omega} |\nabla u_k|^{p(x)} \, dx \leq C \int_{\Omega} \frac{1}{p^+} |\nabla u_k|^{p(x)} \, dx \leq C \int_{\Omega} \frac{1}{p(x)} |\nabla u_k|^{p(x)} \, dx \leq C
\]
The existence and uniqueness of minimizer

we obtain $Q_{p(x)}(u_k) + Q_{p(x)}(\nabla u_k) \leq C$. This implies that $\{u_k\}_{k=1}^\infty$ is a uniformly bounded sequence in $W^{1,p(x)}(\Omega)$. Meanwhile, $\{u_k\}_{k=1}^\infty$ is uniformly bounded in $L^2(\Omega)$. Since $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ is reflexive Banach space, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$, and a function $u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$, such that

$$u_{k_j} \rightharpoonup u \text{ in } W^{1,p(x)}(\Omega) \cap L^2(\Omega)$$

By Lemma 2.4, $E_{p(x)}$ is lower semi continuous in $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Then we have

$$E_{p(x)}(u) \leq \liminf_{j \to \infty} E_{p(x)}(u_{k_j}) = \mu.$$ 

Therefore, $u$ is a minimizer of $E_{p(x)}$. The uniqueness follows from the strict convexity of $E_{p(x)}(u)$ about $u$. 

The existence and uniqueness of minimizer II

**Theorem 2.2** Let $\Omega \subset \mathbb{R}^2$ be bounded open set with Lipschitz boundary, $w \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $f - w \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$. Then the minimization problem

$$
\min_{u \in W^{1,p(x)}(\Omega) \cap L^2(\Omega), \; u-w \in W^{1,p(x)}_0(\Omega)} \left\{ E_{p(x)}(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 \, dx \right\} 
$$

(2.2)

has unique minimizer $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, which satisfies $u - w \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$. 
The existence and uniqueness of minimizer II

Let \( u \in U(= W^{1,p(x)}(\Omega) \cap L^2(\Omega)) \) and denote \( a = \max\{\|w\|_{\infty}, \|f\|_{\infty}\} \). Let \( u_a \) be the function \( u \) which has been cut-off at \( -a \) and \( a \), i.e. \( u_a = \min\{a, \max\{-a, u\}\} \). By definition of \( a \), it is easy to see that \( u_a - w \in W^{1,p(x)}_0(\Omega) \cap L^2(\Omega) \). Moreover,

\[
\nabla u_a = \begin{cases} 
\nabla u, & |u| \leq a \\
0, & |u| > a
\end{cases}
\]

Hence \( |\nabla u_a| \leq |\nabla u| \) a.e. \( x \in \Omega \) and so \( E_{p(x)}(u_a) \leq E_{p(x)}(u) \). It follows that it suffices to look for minimizers in the set \( U_a = \{u_a : u \in U\} \).

Let \( \mu = \inf_{v \in U_a} E_{p(x)}(v) \), and \( \{u_k\}_{k=1}^{\infty} \subset U_a \) be a minimizing sequence. Then \( E_{p(x)}(u_k) \to \mu \). Hence

\[
\int_{\Omega} \frac{1}{p(x)} |\nabla u_k|^{p(x)} \, dx \leq C, \quad \int_{\Omega} (u_k - f)^2 \, dx \leq C.
\]

By the \( W^{1,1} \)-Sobolev-Poincaré inequality, the embedding \( L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega) \) and the fact \( u_k - w \in W^{1,1}_0(\Omega) \), we get
The existence and uniqueness of minimizer II

\[
\int_\Omega |u_k|^{p(x)} \, dx = \int_\Omega |u_k|^{p(x)-1} \, |u_k| \, dx \leq a^{p-1} \int_\Omega |u_k| \, dx
\]
\[
\leq C \int_\Omega |u_k - w| + |w| \, dx \leq C \int_\Omega |\nabla u_k - \nabla w| \, dx + C
\]
\[
\leq C \int_\Omega |\nabla u_k| \, dx + C \leq C \int_\Omega |\nabla u_k|^{p(x)} \, dx + C
\]
\[
\leq C \int_\Omega \frac{1}{p(x)} |\nabla u_k|^{p(x)} \, dx + C
\]

Together with the inequality \( \int_\Omega |\nabla u_k|^{p(x)} \, dx \leq C \), we obtain \( Q_{p(x)}(u_k) + Q_{p(x)}(\nabla u_k) \leq C \), which implies \( \{u_k\}_{k=1}^\infty \) is uniformly bounded in \( W^{1,p(x)}(\Omega) \). Meanwhile, \( \int_\Omega (u_k - f)^2 \, dx \leq C \) results in the uniformly boundedness of \( \{u_k\}_{k=1}^\infty \) in \( L^2(\Omega) \).
The existence and uniqueness of minimizer II

Since $W^{1,p(x)}(\Omega) \cap L^2(\Omega)$ is reflexive Banach space, there exists a subsequence 
\[ \{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}, \] 
and $u \in W^{1,p(x)}(\Omega)$ such that 
\[ u_{k_j} \rightharpoonup u \text{ in } W^{1,p(x)}(\Omega) \cap L^2(\Omega). \]

Moreover, since 
\[ \{u_k\}_{k=1}^{\infty} \subset U_a, \] 
we conclude that $u \in W^{1,p(x)} \cap L^\infty(\Omega)$. We assert next that, $u - w \in W^{1,p(x)}_0 \cap L^\infty(\Omega)$. To see this, note that for $w \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $u_k - w \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$. Since $W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$ is closed, linear subspace of $W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, it is weakly closed. Hence $u - w \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$. Then by Lemma 2.4, 
\[ E_{p(x)}(u) \leq \liminf_{j \to \infty} E_{p(x)}(u_{k_j}) = \mu. \]

Therefore, we conclude that $u$ is the minimizer of $E_{p(x)}$. 
Uniqueness follows from the strictly convexity of $E_{p(x)}(u)$ in $u$. 

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Euler Lagrange equation

(2.1)

\[
\begin{align*}
\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) - \lambda (u - f) &= 0, \quad x \in \Omega \\
\frac{\partial u}{\partial N} &= 0, \quad x \in \partial \Omega
\end{align*}
\]

where $N$ denotes the unit outward normal of $\partial \Omega$. 
Euler Lagrange equation

(2.1)
\[
\begin{align*}
\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) - \lambda (u - f) &= 0, \quad x \in \Omega \\
\frac{\partial u}{\partial N} &= 0, \quad x \in \partial \Omega
\end{align*}
\]

where \( N \) denotes the unit outward normal of \( \partial \Omega \).

(2.2)
\[
\begin{align*}
\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) - \lambda (u - f) &= 0, \quad x \in \Omega \\
u &= w, \quad x \in \partial \Omega
\end{align*}
\]
The associated heat flow

Using steepest descent method, the associated heat flow to problem (2.1) is given by

\[
\begin{aligned}
\left\{
\begin{array}{l}
u_t = \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) - \lambda (u - f), \quad (x, t) \in \Omega^T \\
\frac{\partial u}{\partial N} = 0, \\
u(0) = f.
\end{array}
\right.
\end{aligned}
\]

(3.1)

(3.2)

(3.3)
Definition of weak solution: Motivation

Denote $F(\nabla u, u, x) = \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\lambda}{2} (u - f)^2$. Then (3.1) is equivalent to $u_t = -F'(\nabla u, u, x)$, where $F'(\nabla u, u, x)$ denotes the Gateaux derivative of $F$ about $u$.

Suppose $u$ be a classical solution of (3.1)-(3.3). For each $v \in L^2(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, multiplying (3.1) by $v - u$, integrating over $\Omega$, we have that

$$\int_{\Omega} u_t (v - u) \, dx = \int_{\Omega} -F'(\nabla u, u, x)(v - u) \, dx.$$

From the convexity of $F(\nabla u, u, x)$, we deduce that

$$\int_{\Omega} u_t (v - u) \, dx + E_{p(x)}(v) \geq E_{p(x)}(u) \quad (3.4)$$

Integrating over $[0, s]$, for any $s \in [0, T]$ yields

$$\int_0^s \int_{\Omega} u_t (v - u) \, dx \, dt + \int_0^s E_{p(x)}(v) \, dt \geq \int_0^s E_{p(x)}(u) \, dt. \quad (3.5)$$
Definition of weak solution: Motivation

On the other hand, if (3.5) holds, setting $v = u + \epsilon \varphi$ in (3.5) with $\varphi \in C_0^\infty(\Omega)$, we obtain

$$\int_0^s \int_\Omega u_t \epsilon \varphi \, dx + \int_0^s E_{p(x)}(u + \epsilon \varphi) \, dt \geq \int_0^s E_{p(x)}(u) \, dt$$

which implies $\int_0^s \int_\Omega u_t \epsilon \varphi \, dx \, dt + \int_0^s E_{p(x)}(u + \epsilon \varphi) \, dt$ attains its minimum at $\epsilon = 0$. Hence

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left( \int_0^s \int_\Omega u_t \epsilon \varphi \, dx \, dt + \int_0^s E_{p(x)}(u + \epsilon \varphi) \, dt \right) = 0,$$

that is,

$$\int_0^s \int_\Omega \dot{u} \varphi \, dx \, dt + \int_0^s \int_\Omega F'(\nabla u, u, x) \varphi \, dx = 0.$$

Since $\varphi$ is arbitrary, $\dot{u} + F'(\nabla u, u, x) = 0$. That is to say, if $u$ satisfies (3.5), then $u$ is a weak solution of (3.5) in the sense of distribution. This motivates us to give the following definition.
Definition of weak solution

**Definition** A function $u \in L^2(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, with $\dot{u} \in L^2(\Omega^T)$ is called a weak solution of equations (3.1)-(3.3) if $u(0) = f$, and for all $v \in L^2(0, T; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, for all $s \in [0, T]$, (3.5) holds.
Approximated heat flow

Let \( \varphi(\xi) = \frac{1}{p(x)} |\xi|^{p(x)} \), then the derivative is \( \varphi_r(\xi) = |\xi|^{p(x)-2} \xi \).

Setting

\[
\varphi^\epsilon(\xi) = \frac{1}{p(x)} \left( \sqrt{|\xi|^2 + \epsilon^2} \right)^{p(x)}.
\]

Then

\[
\varphi_r^\epsilon(\xi) = \frac{\xi}{(\sqrt{|\xi|^2 + \epsilon^2})^{2-p(x)}}.
\]

It is easy to see that \( \varphi^\epsilon(\xi) \) is convex in \( \xi \) and \( \varphi^\epsilon \to \varphi \) as \( \epsilon \to 0 \).

Suppose \( 0 < \epsilon < 1 \).
Approximated heat flow

\[
\begin{align*}
\begin{cases}
    u_t &= \epsilon \Delta u + \text{div} (\varphi^\epsilon (\nabla u)) - \lambda (u - f_\delta), \quad (x, t) \in \Omega^T \\
    \frac{\partial u}{\partial N} &= 0, \quad (x, t) \in \partial \Omega^T \\
    u(0) &= f_\delta.
\end{cases}
\end{align*}
\]

where \( f_\delta \in C^\infty(\tilde{\Omega}) \) has the following properties

\[
f_\delta \to f \quad \text{in} \quad L^2(\Omega), \quad \| f_\delta \|_{L^\infty(\Omega)} \leq \| f \|_{L^\infty(\Omega)}, \quad \varphi(\nabla f_\delta) \leq \varphi(\nabla f).
\]
Lemma 3.1  The problem (3.6)-(3.8) has unique weak solution $u_\delta^\epsilon$, with $u_\delta^\epsilon \in L^\infty(0, T; W^{1,p(\cdot)}(\Omega))$ and $\dot{u}_\delta^\epsilon \in L^2(0, T; L^2(\Omega))$ such that

$$\int_0^\infty \int_\Omega |\dot{u}_\delta^\epsilon|^2 \, dx \, dt + \sup_{t > 0} \left\{ \int_{\Omega} \frac{\epsilon}{2} |\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2} (u_\delta^\epsilon - f_\delta)^2 \, dx \right\}$$

$$\leq 2 \left\{ \int_{\Omega} \frac{\epsilon}{2} |\nabla f_\delta|^2 \, dx + \varphi(\nabla f) \, dx + 1 \right\}.$$  \hspace{1cm} (3.11)
Approximated heat flow

Proof: Multiplying (3.6) by $\dot{u}_\delta^\varepsilon$ and integrating on $\Omega$, we get

$$\int_Q |\dot{u}_\delta^\varepsilon|^2 = \int_Q \varepsilon \dot{u}_\delta^\varepsilon \Delta u_\delta^\varepsilon + \int_Q \dot{u}_\delta^\varepsilon \text{div}(\varphi_r^\varepsilon(\nabla u_\delta^\varepsilon)) + \int_Q \dot{u}_\delta^\varepsilon (u_\delta^\varepsilon - f_\delta)$$

Then

$$\int_Q |\dot{u}_\delta^\varepsilon|^2 + \frac{d}{dt} \left\{ \int_Q \frac{\varepsilon}{2} |\nabla u_\delta^\varepsilon|^2 + \varphi^\varepsilon (\nabla u_\delta^\varepsilon) + \frac{\lambda}{2} (u_\delta^\varepsilon - f_\delta)^2 \right\} = 0.$$ 

Integrating the above formula on $(0, s)$,

$$\int_0^s \int_Q |\dot{u}_\delta^\varepsilon|^2 dx + \frac{d}{dt} \left\{ \int_Q \frac{\varepsilon}{2} |\nabla u_\delta^\varepsilon|^2 + \varphi^\varepsilon (\nabla u_\delta^\varepsilon) + \frac{\lambda}{2} (u_\delta^\varepsilon - f_\delta)^2 \right\}$$

$$= \left\{ \int_Q \frac{\varepsilon}{2} |\nabla f_\delta|^2 + \varphi^\varepsilon (\nabla f_\delta) + \frac{\lambda}{2} (f_\delta - f_\delta)^2 \right\}.$$ 

Therefore

$$\int_0^\infty \int_Q |\dot{u}_\delta^\varepsilon|^2 dxd\tau \leq 2 \left\{ \int_Q \frac{\varepsilon}{2} |\nabla f_\delta|^2 + \varphi^\varepsilon (\nabla f_\delta) \right\}.$$ 

$0 < \varepsilon < 1$ yields the conclusion.
Approximated heat flow

Lemma 3.2  Let \( f \in W^{1,p(x)} \cap L^\infty(\Omega) \), and \( u_\delta^\epsilon \) be the weak solution of problem (3.6)-(3.8). Then

\[
\| u_\delta^\epsilon \|_{L^\infty(\Omega^T)} \leq \| f \|_{L^\infty(\Omega)}.
\] (3.12)
Approximated heat flow

Proof: Let $G$ be a truncation function of class $C^1$ such that $G(t) = 0$ on $(-\infty, 0]$, and $G$ is strictly increasing in $[0, +\infty)$, and $G' \leq M$ where $M$ is a constant. Let $k = \|f\|_{L^\infty(\Omega)}$ and set $v = G(u_\epsilon - k)$. Then $\nabla v = G'(u_\epsilon - k) \nabla u_\epsilon$. Multiplying (3.6) by $G$ and integrating over $\Omega$ yields

$$0 = \int_\Omega \dot{u}_\delta G(u_\delta^\epsilon - k) dx + \epsilon \int_\Omega |\nabla u_\delta^\epsilon|^2 G'(u_\delta^\epsilon - k) dx$$
$$+ \int_\Omega \varphi_\epsilon(\nabla u_\delta^\epsilon) \nabla u_\delta^\epsilon G'(u_\delta^\epsilon - k) dx + \lambda \int_\Omega (u_\delta^\epsilon - f_\delta) G(u_\delta^\epsilon - k) dx \quad (3.13)$$

By the definition of $\varphi$, $\int_\Omega \varphi_\epsilon(\nabla u_\delta^\epsilon) \nabla u_\delta^\epsilon G'(u_\delta^\epsilon - k) dx \geq 0$. It is obvious that $\epsilon \int_\Omega |\nabla u_\delta^\epsilon|^2 G'(u_\delta^\epsilon - k) dx \geq 0$. If $\int_\Omega (u_\delta^\epsilon - f_\delta) G(u_\delta^\epsilon - k) dx \leq 0$, then we get $u_\delta^\epsilon \leq \|f_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} = k$. Otherwise, we have $\int_\Omega (u_\delta^\epsilon - f_\delta) G(u_\delta^\epsilon - k) dx \geq 0$. Hence (3.13) yields

$$\int_\Omega \dot{u}_\delta G(u_\delta^\epsilon - k) dx \leq 0.$$

Since $0 \leq G' \leq M$ we deduce that

$$\frac{d}{dt} \int_\Omega (G(u_\delta^\epsilon - k))^2 dx \leq 0.$$

Therefore $\int_\Omega (G(u_\delta^\epsilon - k))^2 dx$ is monotonically decreasing function about $t$ and then

$$\int_\Omega (G(u_\delta^\epsilon - k))^2 dx \leq \int_\Omega (G(u_\delta^\epsilon - k))^2 dx|_{t=0} = \int_\Omega (G(f_\delta - k))^2 dx = 0.$$

So we have proved that $u_\delta^\epsilon \leq k$. Similarly, $u_\delta^\epsilon \geq -k$ can be proved.
Existence and uniqueness of solution

**Theorem 3.1** Suppose $f \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Then (3.1)-(3.3) has a unique weak solution $u \in L^\infty(0, T; W^{1,p(x)}(\Omega) \cap L^\infty(\Omega))$, with $\dot{u} \in L^2(\Omega^T)$. 
Existence of solution

Proof: Let \( \{u_{\delta}^\varepsilon\} \) be the sequence of solution to (3.6)-(3.8). By (3.11) and (3.12), we get that \( \{u_{\delta}^\varepsilon\} \) has uniformly bounded \( L^\infty(\Omega^\infty) \) norm about \( \varepsilon \), and \( \{\dot{u}_{\delta}^\varepsilon\} \) has uniformly bounded \( L^2(\Omega^\infty) \) norm. Then there exists a subsequence, also denoted by \( \{u_{\delta}^\varepsilon\} \), and a function \( u_{\delta} \in L^\infty(\Omega^\infty) \), such that as \( \varepsilon \to 0 \),

\[
\begin{align*}
\lim_{\varepsilon \to 0} u_{\delta}^\varepsilon &\quad \text{in} \quad L^\infty(\Omega^\infty), \quad \text{(3.14)} \\
\lim_{\varepsilon \to 0} \dot{u}_{\delta}^\varepsilon &\quad \text{in} \quad L^2(\Omega^\infty). \quad \text{(3.15)}
\end{align*}
\]

The same argument used in the proof of Lemma 3.1 [10] gives us that \( \dot{u}_{\delta} = w, \ u_{\delta}(0) = f_{\delta} \). Then we have \( \dot{u}_{\delta} \in L^2(\Omega^\infty) \). Moreover, for all \( \phi \in L^2(\Omega) \),

\[
\begin{align*}
\int_{\Omega} (u_{\delta}^\varepsilon(\cdot, t) - f_{\delta}) \phi(x) dx &= \int_0^t \int_{\Omega} \dot{u}_{\delta}^\varepsilon(x, s) 1_{[0,t]}(s) \phi(x) dxds \\
\lim_{\varepsilon \to 0} \int_0^t \int_{\Omega} \dot{u}_{\delta}(x, s) 1_{[0,t]}(s) \phi(x) dxdt &= \int_{\Omega} (u_{\delta}(\cdot, t) - f_{\delta}) \phi(x) dx (\varepsilon \to 0).
\end{align*}
\]

which implies that

\[
u_{\delta}^\varepsilon(\cdot, t) \rightharpoonup u_{\delta}(\cdot, t) \text{ weakly in } L^2(\Omega).
\]

From (3.11), for each \( t > 0 \), \( \{u_{\delta}^\varepsilon(\cdot, t)\} \) is a uniformly bounded sequence in \( W^{1,1}(\Omega) \). Then there exists a subsequence, also denoted by \( \{u_{\delta}^\varepsilon(\cdot, t)\} \), such that

\[
u_{\delta}^\varepsilon(\cdot, t) \to u_{\delta}(\cdot, t) \text{ in } L^1(\Omega). \quad \text{(3.16)}
\]
Existence of solution

From (3.12), (3.14) and (3.16), we obtain

\[
\int_{\Omega} |u_\delta^\epsilon(\cdot, t) - u_\delta(\cdot, t)|^2 \, dx \\
\leq \|u_\delta^\epsilon(\cdot, t) - u_\delta(\cdot, t)\|_{L^\infty(\Omega)} \int_{\Omega} |u_\delta^\epsilon(\cdot, t) - u_\delta(\cdot, t)| \, dx \\
\leq C \|f\|_{L^\infty(\Omega)} \int_{\Omega} |u_\delta^\epsilon(\cdot, t) - u_\delta(\cdot, t)| \, dx dt \to 0 \quad (\text{as } \epsilon \to 0).
\]

Therefore,

\[ u_\delta^\epsilon(\cdot, t) \to u_\delta(\cdot, t) \text{ in } L^2(\Omega). \tag{3.17} \]

For any \( v \in L^2((0, \infty); H^1(\Omega)) \), Multiplying (3.6)(where \( u \) is replaced by \( u_\delta^\epsilon \)) by \( (v - u_\delta^\epsilon) \), using the convexity of \( \varphi^\epsilon \) in \( r \), we get

\[
\int_0^s \int_{\Omega} u_\delta^\epsilon(\cdot, \tau) + \frac{\epsilon}{2} |\nabla v|^2 + \varphi^\epsilon(\nabla v) + \frac{\lambda}{2} (v - f_\delta)^2 \, dx dt \\
\geq \int_0^s \int_{\Omega} \frac{\epsilon}{2} |\nabla u_\delta^\epsilon|^2 + \varphi^\epsilon(\nabla u_\delta^\epsilon) + \frac{\lambda}{2} (u_\delta^\epsilon - f_\delta)^2 \, dx dt
\]

From (3.15), (3.17) and the lower semi-continuity of \( \varphi^\epsilon \) we obtain
Existence of solution

\[ \int_0^s \int_{\Omega} \dot{u}_\delta^\epsilon (v - u_\delta^\epsilon) + \varphi^\epsilon (\nabla v) + \frac{\lambda}{2} (v - f_\delta)^2 \, dxdt \geq \liminf_{\epsilon \to 0} \int_0^s \int_{\Omega} \varphi^\epsilon (\nabla u_\delta^\epsilon) + \frac{\lambda}{2} (u_\delta^\epsilon - f_\delta)^2 \, dxdt. \]

Let \( \epsilon \to 0 \) yields

\[ \int_0^s \int_{\Omega} \dot{u}_\delta (v - u_\delta) + \varphi (\nabla v) + \frac{\lambda}{2} (v - f_\delta)^2 \, dxdt \geq \int_0^s \int_{\Omega} \varphi (\nabla u_\delta) + \frac{\lambda}{2} (u_\delta - f_\delta)^2 \, dxdt \quad (3.18) \]

holds for any \( v \in L^2(0, \infty; H^1(\Omega)) \). By approximation, (3.18) still holds for any \( v \in L^2(0, \infty; W^{1,p(x)}(\Omega) \cap L^2(\Omega)) \).

It remains to pass to a limit as \( \delta \to 0 \). In (3.11) we let \( \epsilon \to 0 \) to get that

\[ \int_0^\infty \int_{\Omega} |\dot{u}_\delta| \, dx \, dt + \sup_{t > 0} \int_{\Omega} \left\{ \varphi (\nabla u_\delta) + \frac{\lambda}{2} (u_\delta - f_\delta)^2 \, dx \right\} \leq C \]

where \( C \) depends on \( f \). Therefore \( \{u_\delta\} \) is uniformly bounded in \( L^\infty (0, \infty; W^{1,p(x)}(\Omega) \cap L^2(\Omega)) \), and then uniformly bounded in \( W^{1,1}(\Omega) \) and also we have \( \dot{u}_\delta \) is uniformly bounded in \( L^2(\Omega^\infty) \).
Existence of solution

Moreover, letting $\epsilon \to 0$ in (3.12) then yields

$$\|u_\delta\|_{L^\infty(\Omega^\infty)} \leq \|f\|_{L^\infty(\Omega)}.$$ 

Hence $\{u_\delta\}$ is uniformly bounded in $L^\infty(\Omega^\infty)$. By the same argument used to get (3.14), (3.15) and (3.17), there exists a subsequence, also denoted by $\{u_\delta\}$ and a function $u \in L^\infty(0, \infty; W^{1,p(x)}(\Omega) \cap L^2(\Omega))$, $\dot{u} \in L^2(\Omega^\infty)$ such that as $\delta \to 0$,

$$u_\delta \rightharpoonup u \text{ weakly* in } L^\infty(\Omega^\infty) \quad (3.19)$$

$$\dot{u}_\delta \rightharpoonup \dot{u} \text{ in } L^2(\Omega^\infty) \quad (3.20)$$

$$u_\delta(\cdot, t) \to u(\cdot, t) \in L^2(\Omega^\infty) \quad (\forall \ t > 0). \quad (3.21)$$

Using the lower semi-continuity of $\varphi$ and (3.19)-(3.21), and letting $\delta \to 0$ in (3.18), we conclude that for all $v \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$

$$\int_0^s \int_\Omega u_t(v - u)dxdt + \int_0^s E_{p(x)}(v)dt \geq \int_0^s E_{p(x)}(u)dt.$$ 

By definition, $u$ is the weak solution of problem (3.1)-(3.3).
Stability

**Theorem 3.2** Assume \( u_1 \) and \( u_2 \) are both weak solution of (3.1)-(3.3), with initial value \( f_1, f_2 \in W^{1,p(x)}(\Omega) \cap L^2(\Omega) \). Then for any \( t > 0 \),

\[
\| u_1 - u_2 \|_{L^\infty(\Omega)} \leq \| f_1 - f_2 \|_{L^\infty(\Omega)}
\]
Stability

Proof Set $k = \|u_1 - u_2\|_{L^\infty(\Omega)}$. Define

$$\begin{cases}
v = u_1 - (u_1 - u_2 - k)_+ \\
w = u_2 + (u_1 - u_2 - k)_+
\end{cases}$$

$$\nabla v = \begin{cases}
\nabla u_1, & u_1 - u_2 \leq k \\
\nabla u_2, & u_1 - u_2 \geq k
\end{cases}
\quad \nabla w = \begin{cases}
\nabla u_2, & u_1 - u_2 \leq k \\
\nabla u_1, & u_1 - u_2 \geq k
\end{cases}$$

For all $t > 0$, we have

$$\int_\Omega \dot{u}_1 (v - u_1) + \varphi(\nabla v) + \frac{\lambda}{2} (v - f_1)^2 \, dx \, dt \geq \int_\Omega \varphi(\nabla u_1) + \frac{\lambda}{2} (u_1 - f_1)^2 \, dx \, dt,$$

$$\int_\Omega \dot{u}_2 (w - u_2) + \varphi(\nabla w) + \frac{\lambda}{2} (w - f_2)^2 \, dx \, dt \geq \int_\Omega \varphi(\nabla u_2) + \frac{\lambda}{2} (u_2 - f_2)^2 \, dx \, dt.$$

Taking summation yields

$$\int_\Omega \dot{u}_1 (v - u_1) + \dot{u}_2 (w - u_2) + \varphi(\nabla v) + \varphi(\nabla w) + \frac{\lambda}{2} (v - f_1)^2 + \frac{\lambda}{2} (w - f_2)^2 \, dx \, dt$$

$$\geq \int_\Omega \varphi(\nabla u_1) + \varphi(\nabla u_2) + \frac{\lambda}{2} (u_1 - f_1)^2 + \frac{\lambda}{2} (u_2 - f_2)^2 \, dx \, dt.$$
Stability

By the definition of \( v, w \), it is clear that

\[
\varphi(\nabla v) + \varphi(\nabla w) = \varphi(\nabla u_1) + \varphi(\nabla u_2),
\]

and

\[
\int_\Omega (u_1 - f_1)^2 + (u_2 - f_2)^2 - (v - f_1)^2 - (w - f_2)^2 dx \\
= \int_\Omega (u_1 - f_1)(u_1 + v - 2f_1) + (u_2 - w)(u_2 + w - 2f_2) dx \\
= \int_\Omega (u_1 - u_2 - k)_+ (2u_1 - 2f_1 - 2u_2 + 2f_2 - 2(u_1 - u_2 - k)_+) dx \\
= \int_\Omega 2(u_1 - u_2 - k)_+ ((u_1 - u_2 - k) - (u_1 - u_2 - k)_+ - (f_1 - f_2 - k)) dx
\]

By definition of \( k \), we have \( f_1 - f_2 - k \leq 0 \). If \( (u_1 - u_2 - k)_+ = 0 \), then

\[
\int_\Omega (u_1 - u_2 - k)_+ ((u_1 - u_2 - k) - (u_1 - u_2 - k)_+ - (f_1 - f_2 - k)) dx = 0
\]

If \( (u_1 - u_2 - k)_+ > 0 \), then
Stability

\[
\int_{\Omega} (u_1 - u_2 - k)^+ ((u_1 - u_2 - k) - (u_1 - u_2 - k)^+ - (f_1 - f_2 - k)) \, dx \\
\geq \int_{\Omega} (u_1 - u_2 - k)^+ ((u_1 - u_2 - k) - (u_1 - u_2 - k)) \, dx = 0.
\]

So we get that

\[
\int_{\Omega} (v - f_1)^2 + (w - f_2)^2 \, dx \leq \int_{\Omega} (u_1 - f_1)^2 + (u_2 - f_2)^2 \, dx.
\]

\[
\Rightarrow \int_{\Omega} u_1 (v - u_1) + u_2 (w - u_2) \, dx \, dt \geq 0.
\]

By definition of \( v \) and \( w \), we get

\[
\int_{\Omega} (u_1 - u_2)(u_1 - u_2 - k)^+ \, dx \leq 0 \Rightarrow \frac{d}{dt} \int_{\Omega} |(u_1 - u_2 - k)^+|^2 \, dx \leq 0.
\]

So \( \int_{\Omega} |(u_1 - u_2 - k)^+|^2 \, dx \) is monotonically decreasing function about \( t \), then

\[
\int_{\Omega} |(u_1 - u_2 - k)^+|^2 \, dx \leq \int_{\Omega} |(f_1 - f_2 - k)^+|^2 \, dx (= 0).
\]

Therefore \( u_1 - u_2 \leq k \). Similarly we can prove \( u_1 - u_2 \geq -k \).
Behavior as $t \rightarrow \infty$

**Theorem 3.3** Let $f \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Then as $t \rightarrow \infty$, the weak solution $u(x, t)$ to (3.1)-(3.3) converge strongly to the solution of the minimization problem (1.4) in $L^2(\Omega)$. 
Behavior as $t \to \infty$

Proof By definition of weak solution (3.5), for any $s > 0$, for any $v(x) \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$, we have

$$\int_0^s \int_{\Omega} \dot{u}(x, t)(v(x) - u(x, t))dxdt + \int_0^s E_{p(x)}(v(x))dt \geq \int_0^s E_{p(x)}(u(x, t))dt,$$

that is,

$$\int_{\Omega} (u(x, s) - f(x))v(x)dx + \frac{1}{2} \int_{\Omega} (u^2(x, s) - f^2(x))dx + s \int_{\Omega} \varphi(\nabla v(x))dx + s \lambda \int_{\Omega} (v(x) - f(x))^2 dx \geq \int_0^s \int_{\Omega} \varphi(\nabla u)dxdt + \int_0^s \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dxdt. (3.22)$$

Define

$$w(x, s) = \frac{1}{s} \int_0^s u(x, t)dt.$$
Behavior as $t \to \infty$

Since $u \in L^\infty(0, \infty; W^{1,p(x)}(\Omega) \cap L^\infty(\Omega))$ for any $s > 0$, we have $w(\cdot, s)$ is uniformly bounded in $W^{1,p(x)}(\Omega)$ and $L^\infty(\Omega)$. Then there exists a subsequence, also denoted by $\{w(x, s)\}$, and a function $\tilde{u} \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, such that

$$w(x, s) \rightharpoonup \tilde{u} \quad \text{in} \quad W^{1,p(x)}(\Omega)$$

$$w(x, s) \to \tilde{u} \quad \text{in} \quad L^2(\Omega)$$

Dividing (3.22) by $s$, then letting $s \to \infty$, we obtain

$$E_{p(x)}(v) \geq E_{p(x)}(\tilde{u})$$

for all $v(x) \in W^{1,p(x)}(\Omega) \cap L^2(\Omega)$. Hence, $\tilde{u}$ is the solution to problem (1.4).
Finite difference scheme

Finite difference scheme \( n = 2 \). Image size is \( N \times N \). \( \tau \) is time step. \( h = 1 \) is space step. Let \( x_i = ih, \; y_j = jh, \; i, j = 0, 1, ..., N, \; t_n = n\tau, n = 0, 1, ... \). Let \( u_{i,j}^n = u(x_i, y_j, t_n), \; u_{ij}^0 = f(x_i, y_j) \). Define

\[
D_x^\pm(u_{i,j}) = \pm[u_{i+1,j} - u_{i,j}], \quad D_y^\pm(u_{i,j}) = \pm[u_{i,j+1} - u_{i,j}],
\]

\[
|D_x(u_{i,j})| = \sqrt{(D_x^+(u_{i,j}))^2 + (m[D_y^+(u_{i,j}), D_y^-(u_{i,j})])^2 + 0.001},
\]

\[
|D_y(u_{i,j})| = \sqrt{(D_y^+(u_{i,j}))^2 + (m[D_x^+(u_{i,j}), D_x^-(u_{i,j})])^2 + 0.001},
\]

where \( m[a, b] = \left(\frac{\text{sign } a + \text{sign } b}{2}\right) \cdot \min(|a|, |b|) \). Then the finite difference scheme of the heat flow (3.1)-(3.3) is given by

\[
u^{k+1} = u^k + \tau \left(D_x^- \left(\frac{D_x^+ u^k}{|D_x u^k|^{1-g}}\right) + D_y^- \left(\frac{D_y^+ u^k}{|D_y u^k|^{1-g}}\right) - \lambda(u^k - f)\right),
\]
Experimental results

(a) True

(b) Noisy
Experimental results

Figure: Comparison of the proposed model and ROF model. (a) The true Lena image; (b) Noisy image (SNR=2.1dB); (c) Restoration result by ROF model; (d) Restoration result by the proposed model.
Experimental results

Figure: Comparison of the proposed model and ROF model in processing piecewise smooth region. Area 1 in each image is zoomed in to show this. (a) The true Lena image; (b) Noisy image; (c) Restoration result by ROF model; (d) Restoration result by the proposed model.
Experimental results

Figure: Comparison of the proposed model and ROF model in processing piecewise smooth region. Area 1 in each image is zoomed in to show this. (a) The true Lena image; (b) Noisy image; (c) Restoration result by ROF model; (d) Restoration result by the proposed model.