Source Reconstruction for 3D Bioluminescence Tomography with Sparse regularization

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Outline

- Problem Description
- Reconstruction based on Gaussian Noise model
- Reconstruction based on Poisson Noise model
- Conclusions and future work

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2. X. Zhang, Y. Lu, T. F. Chan, *A novel sparsity reconstruction method from Poisson data for bioluminescence tomography*, submit
Problem description

Bioluminescent Imaging system

- Powerful tumor detection/monitoring technique for *in vivo* Imaging. Extremely low backgrounds, high sensitivity and relatively simple instrumentation.

- Tissue is a medium exhibiting both scattering and absorption properties. Amount of light is proportional to number of cells producing it.

- BLT reconstruction: reconstruct sources from observed photons intensity on the surfaces.

Surface intensity depends on:
- Source depth
- Source shape and brightness
- Surface shape (curvature)
- Wavelength
- Tissue optical properties

*Figure: By the Courtesy of XENOGEN*
Photons Diffusion Model

For \( x \in \Omega \subset \mathbb{R}^3 \), \( \lambda \): wavelength, let

- \( \Phi(x, \lambda) \): photon flux density
- \( S(x, \lambda) \): source energy density

Steady-state diffusion equation

\[
\nabla \cdot \left( \gamma \nabla \Phi \right) - \mu_a \Phi + S = 0, \quad (\forall x \in \Omega)
\]

where \( \gamma(x, \lambda) \) is diffusion coefficient and \( \mu_a(x, \lambda) \) is absorption coefficient

Robin boundary condition

\[
\Phi + c \gamma(v(x) \cdot \nabla \Phi) = 0 \quad (x \in \partial \Omega)
\]

where \( v \): unit outer normal on \( \partial \Omega \); \( c \): parameter

Measurements: the photon density on the body surface \( \partial \Omega \) for discretized \( \lambda_i \)

\[
Q(x, \lambda_i) = \frac{\Phi(x, \lambda_i)}{c} = -\gamma(v \cdot \nabla \Phi(x, \lambda_i)) \quad (x \in \partial \Omega)
\]
Inverse Problem: Linear Relationship establishment

- Weak solution for $\Phi(x, \lambda)$
- Using finite elements method to discretize the domain $\Omega$, $\Phi(x, \lambda)$ and $S(x, \lambda)$
- Linear equation

$$Au = f$$

where $u$: unknown nodes vector for $S$ in $\Omega$, $f$ is the measurable nodes vector for $\Phi$ on the boundary $\partial\Omega$.

- $A$ is undetermined, highly ill-posed, depending on mesh sizes and shape functions.
Noisy Model

- **Gaussian Noise Model**
  \[ f = Au + N \]
  \( N \): noise modeled as a Gaussian distribution

- **Poisson Noise Model**
  \[ f_i = \text{Poisson}((Au)_i) \]
  \[ P(f_i|(Au)_i) = \frac{(Au)_i^{f_i} e^{(Au)_i}}{f_i!} \]
Classical Methods for Gaussian noise model

- Classical Maximum Likelihood method (Least square)

\[
\min_u \frac{1}{2} \|Au - f\|^2 \quad \text{s. t. } D = \{0 \leq u \leq C\}
\]

- Tikhonov regularization

\[
\min_u \frac{1}{2} \|Au - f\|^2 + \frac{\delta}{2} \|u\|^2 \quad \text{s. t. } D = \{0 \leq u < C\}
\]
Sparsity as a priori information

- $l^1$ regularization: **sparsity**

$$\min_S \frac{1}{2} \| Au - f \|^2 + \mu \| u \|_1 \quad \text{s.t. } D = \{0 \leq u \leq C\}$$

where $\| S \|_1 = \sum_i |S_i|$ denotes the $l^1$ norm of the vector $S$.

- Algorithm: Bound constrained quasi-Newton method (BLMVM)$^3$ using smoothed $l^1$ norm.

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Simulations settings

- Data simulated by Monte Carlo
- Simulation of photon diffusion
- Cube domain with a width of $15mm \times 15mm \times 15mm$ and discretized by hexahedra based FEM
- Photon distribution observed only on the top surface of the cubic domain at three wavelengths $\lambda_1 = 600nm, \lambda_2 = 650nm, \lambda_3 = 700nm$
Gaussian Noise model

**Observed photon distribution on one surface**

<table>
<thead>
<tr>
<th>Photons</th>
<th>600nm</th>
<th>650nm</th>
<th>700nm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>$10^4$</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
</tbody>
</table>

**Figure:** Observed photon distribution of single source (located at (0, 0, 0)) at the top surface.
Single Source Reconstruction

Figure: Reconstruction for single source located at $(0, 0, 0)$. First row: $10^6$ photons; Second row: $10^4$ photons
Dual sources with homogeneous media

Figure: Dual source BLT reconstructions when the real source central positions are at \((-3.0, 0.0, 3.0)\) and \((3.0, 0.0, 3.0)\) with $10^4$ photons for 600 nm.
Dual sources with heterogeneous media

**Figure:** Dual source BLT reconstructions when the real source central positions are at $(−3.0, 0.0, 3.0)$ and $(3.0, 0.0, 3.0)$ with $10^4$ photons for $600\,nm$. 

Gaussian Noise model
**Experimental data Reconstruction**

GFP filter observation  |  DsRed filter observation  |  Volumetric mesh

No regularization  |  $l^2$ regularization  |  $l^1$ regularization

**Figure:** Experimental BLT reconstructions with mouse-shaped phantom. The actual source position is at $(114.5, 131.0, 3.0)$ (CT scanning), No regularization: $(111.7, 132.6, 2.7)$, $l^2$ regularization: $(115.1, 131.7, 2.4)$, $l^1$ regularization: $(114.7, 131.7, 2.9)$. 
Poisson Noise model

- Random observation: $f_i \sim \text{Poisson}(Au)_i$
- Maximum a-posteriori (MAP) estimation

\[
\max_{u \geq 0} p(u|f) \Rightarrow \min_{u \geq 0} \sum_{\Omega}(Au - f \log Au)_i
\]

\[
\Rightarrow \min_{u \geq 0} D_{KL}(f, Au)
\]

where $D_{KL}$ is the Kullback-Leibbler distance.

- Optimality condition KKT:
  \[
  \begin{cases}
  \nabla F(u) - \lambda = 0 \\
  \lambda_i u_i = 0 & \text{for } i = 1, \ldots, m \\
  \lambda_i \geq 0 & \text{for } i = 1, \ldots, m
  \end{cases}
  \]

\[(1)\]

where $\nabla F(u) = A^*1 - A^*(\frac{f}{Au})$
Reconstruction models

- EM algorithm (Richardson-Lucy algorithm): fixed point algorithm on

\[ (A^* \mathbf{1} - A^*(\frac{f}{Au}))u = 0 \]  \hspace{1cm} (2)

\[ u_{k+1} = u_k A^* \frac{A^* \mathbf{1}}{A u_k} (\frac{f}{Au_k}) \]

- Sparsity A priori: \( l^1 \) norm regularization:

\[ \min_{u \geq 0} \Phi(u) = D_{KL}(f, Au) + \mu \|u\|_1 \]

- \( l^0 \) norm regularization:

\[ \min_{u \geq 0} D_{KL}(f, Au) \quad \text{s.t} \quad 0 < \|u\|_0 \leq K \]
Poisson $\ell^1$ minimization algorithm

- Forward backward splitting + EM-$\ell^1$ Algorithm \(^4\)

\[
\begin{align*}
\mathbf{u}^{k+\frac{1}{2}} &= (1 - \omega_k) \mathbf{u}^k + \omega_k \mathbf{u}^k \frac{A^*}{A^* \mathbf{1}_\Omega} \left( \frac{\mathbf{f}}{A \mathbf{u}^k} \right) \\
\mathbf{u}^{k+1} &= \arg\min_{\mathbf{u} \geq 0} \frac{1}{2} \sum_{\Omega} \frac{(A^* \mathbf{1}_\Omega)_i}{(\mathbf{u}^k)_i} (\mathbf{u} - \mathbf{u}^{k+\frac{1}{2}})_i^2 + \omega_k \mu \| \mathbf{u} \|_1
\end{align*}
\]

\(^4\)Brune, Sawatzky, Wubbeling, Lusters, Burger, 2009
**SPIRAL-TAP**

Let $F(u) = D_{KL}(f, Au)$, 
$\tilde{F}_k(u) = F(u^k) + (u - u^k)^T \nabla F(u^k) + \frac{\alpha_k}{2} \|u - u^k\|^2$. 

$$u^{k+1} = \arg\min_{u \geq 0} \tilde{F}_k(u) + \mu \|u\|_1$$

**SPIRAL-TAP**

- Initialize Choose $\eta > 1, \sigma \in (0, 1), M \in \mathbb{Z}^+, 0 < \alpha \in (\alpha_{\text{min}}, \alpha_{\text{max}})$, and initial solution $u_0$. Start iteration counter $k = 0$.
- Repeat
  - choose $\alpha \in (\alpha_{\text{min}}, \alpha_{\text{max}})$ by Barzilai-Borwein method:
    $$\alpha_k = \frac{(u^k - u^{k-1})^T \nabla^2 F(u^k)(u^k - u^{k-1})}{\|u^k - u^{k-1}\|^2}$$
  - $u_{k+1} = \arg\min_{u \geq 0} \frac{1}{2} \|u - s^k\| + \frac{\mu}{\alpha_k} \|u\|_1$
  - $\alpha_k \leftarrow \eta \alpha_k$, until $u_{k+1}$ satisfies acceptance criteria:
    $$\Phi(u^{k+1}) \leq \max[k-M]_+, \ldots, k \Phi(u^k) - \frac{\sigma \alpha_k}{2} \|u^{k+1} - u^k\|^2$$
- $k \leftarrow k + 1$, until stopping criterion is satisfied

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5 Harmany, Marcia, Willett, This is SPIRAL-TAP: Sparse Poisson Intensity Reconstruction Algorithms-Theory and Practice, preprint
**\( l^0 \)** minimization with Orthogonal Matching Pursuit (OMP)

Gaussian Noise model:

\[
\min_u \|A u - f\|^2 \quad \text{s. t.} \quad 0 < |u|_0 \leq K
\]

**OMP algorithm**

1. Initialize the residual \( r_0 = f \), the index set \( \Gamma_0 = [\ ] \);
2. At step \( k \), find the index \( j \) that solves the easy optimization problem
   \[
   j_k = \arg \max_{i=1,...,n} |\langle r_k, A_j \rangle|
   \]
3. Add in the index set \( \Gamma_{k+1} = \Gamma_k \cup \{j_k\} \)
4. Solve a least-squares problem to obtain a new signal estimate:
   \[
   u_{k+1} = \arg \min_u \|A|_{\Gamma_{k+1}} u - f\|^2
   \]
5. Repeat (2) until \( k = K \).
Iterative Poisson OMP Algorithm

\[
\min_{u \geq 0} D_{KL}(Au, f) \quad \text{s. t.} \quad 0 < |u|_0 \leq K
\]

**Iterative Poisson OMP**

- Initialize the index set \( \Gamma_0 = [\ ] \), the set of all index \( I = \{1, \cdots, n\} \);
- Outer iteration: \( t = 0 \) to \( t = T \)
- Set \( \hat{\Gamma}_0 = \Gamma_t \)
  - Inner iteration: \( k = 0 \) to \( k = K - 1 \)
  - Find \( \hat{u}_{k+1} \) and \( i_{k+1} \) by solving the subproblem:
    \[
    [i_{k+1}, \hat{u}_{k+1}] = \arg \min_{i \in \{1, \cdots, n\}} \min_{u \geq 0} D_{KL}(f, A|\hat{\Gamma}_k \cup \{i\}u)
    \]
  - Merge the new index: \( \hat{\Gamma}_{k+1} = \hat{\Gamma}_k \cup \{i_{k+1}\} \).
- Find the \( K \) largest elements of \( \hat{u}_K \) and set the corresponding index set as \( \Gamma_{t+1} \).
- If \( \Gamma_{t+1} \) is the same as \( \Gamma_t \), stop; otherwise continue.
## Compressive sensing simulations

<table>
<thead>
<tr>
<th>Tests ( (m, n, d, K) )</th>
<th>Photons</th>
<th>EM-L1</th>
<th>SPIRAL</th>
<th>IterPOMP</th>
</tr>
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<tr>
<td>(256, 1024, 40, 4)</td>
<td>1e6</td>
<td>19.59</td>
<td>19.59</td>
<td>19.59</td>
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<td>4.59e-3</td>
<td>4.61e-3</td>
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<td></td>
<td>0.02</td>
<td>0.01</td>
<td>3.00</td>
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<tr>
<td></td>
<td>1e4</td>
<td>22.84</td>
<td>22.84</td>
<td>22.87</td>
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<td>4.70e-2</td>
<td>4.39e-2</td>
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<tr>
<td></td>
<td></td>
<td>0.02</td>
<td>0.02</td>
<td>2.35</td>
</tr>
<tr>
<td>(256, 1024, 400, 4)</td>
<td>1e6</td>
<td>109.32</td>
<td>109.33</td>
<td>109.32</td>
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<tr>
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<td>2.39e-3</td>
<td>2.36e-3</td>
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<tr>
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<td>0.23</td>
<td>0.07</td>
<td>3.00</td>
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<td>103.66</td>
<td>103.66</td>
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<td>2.55e-2</td>
<td>2.55e-2</td>
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<td>0.20</td>
<td>0.06</td>
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<td>(256, 1024, 40, 10)</td>
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<td>1.31</td>
<td>0.01</td>
<td>5.99</td>
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<td></td>
<td>1e4</td>
<td>35.36</td>
<td>35.36</td>
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<td>5.48e-2</td>
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<td>0.10</td>
<td>0.03</td>
<td>5.92</td>
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</table>

**Table:** Compressive sensing reconstruction based on 5 trials. For each test setting, we generate a Bernoulli matrix of size \( m \times n \) with \( d \) non zeros in each row. The randomly generated signals have \( K \) nonzeros of intensity \( 1e6 \) or \( 1e4 \).
BLT Reconstruction results with FEM simulated data

True

EM-$\ell^1$

SPIRAL

POMP

IterPOMP
Dual sources

- True
- EM-$\ell^1$
- SPIRAL
- POMP
- IterPOMP
More sources setting tests

<table>
<thead>
<tr>
<th>True Sources</th>
<th>Reconstructed Sources</th>
<th>RelErr</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5, 0.5, 0.5, 5.0e+6)</td>
<td>(0.5, 0.5, 0.5, 5.0e+6)</td>
<td>5.50e-4</td>
<td>3651.7</td>
</tr>
<tr>
<td>(3.5, -3.5, 2.5, 3.0e+6)</td>
<td>(3.5, -3.5, 2.5, 3.0e+6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-2.5, 2.5, 4.5, 1.0e+6)</td>
<td>(-2.5, 2.5, 4.5, 1.0e+6)</td>
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<td></td>
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<tr>
<td>(2.5, 1.5, 5.5, 2.0e+6)</td>
<td>(2.5, 1.5, 5.5, 2.0e+6)</td>
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<td>(0.5, 0.5, 0.5, 5.0e+6)</td>
<td>8.52e-1</td>
<td>593.6</td>
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<tr>
<td>(3.5, -3.5, 2.5, 3.0e+6)</td>
<td>(3.5, -3.5, 2.5, 3.0e+6)</td>
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<tr>
<td>(0.5, 0.5, 2.5, 2.0e+6)</td>
<td>(0.5, 0.5, 1.5, 4.4e+6)</td>
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<tr>
<td>(-2.5, 2.5, 4.5, 1.0e+6)</td>
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<td>(0.5, 0.5, 0.5, 5.0e+4)</td>
<td>(0.5, 0.5, 0.5, 5.0e+4)</td>
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<td>(-2.5, 2.5, 4.5, 1.0e+4)</td>
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</table>

Table: Synthesized test examples with iterPOMP algorithm for $\ell^0$ regularization model
<table>
<thead>
<tr>
<th>Test with larger $K$</th>
<th>True Sources</th>
<th>Reconstructed Sources</th>
<th>RelErr</th>
<th>Time</th>
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<tbody>
<tr>
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<td>(0.5, 0.5, -0.5, 1.2e+6)</td>
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<td>(-2.5, 2.5, 4.5, 1.0e+4)</td>
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<td>1.21e+0</td>
<td>770.0</td>
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</table>
Monte-Carlo simulation recovery

**Figure:** Triple sources with MC generated data. Left: true sources. Right: IterPOMP Recovered. True location and intensity: (-2.5, 2.5, 4.5, 1e+4), (0.5, 0.5, 0.5, 5e+4), (3.5, -3.5, 2.5, 3e+4). The $\ell^0$ algorithm has faithfully reconstructed the locations.
Figure: Triple sources with MC generated data. True location and intensity: $(-2.5, 2.5, 4.5, 1e+4)$, $(0.5, 0.5, 0.5, 5e+4)$, $(3.5, -3.5, 2.5, 3e+4)$. The $\ell^0$ algorithm has faithfully reconstructed the locations.
More test

<table>
<thead>
<tr>
<th>True sources</th>
<th>Reconstructed</th>
<th>RelErr</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5, 0.5, 0.5, 5.0e+6)</td>
<td>(0.5, 0.5, 0.5, 1.7e+6)</td>
<td>6.64e-1</td>
<td>339.3</td>
</tr>
<tr>
<td>(3.5, -3.5, 2.5, 3.0e+6)</td>
<td>(3.5, -3.5, 2.5, 1.0e+6)</td>
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<td></td>
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<tr>
<td>(-2.5, 2.5, 4.5, 1.0e+6)</td>
<td>(-2.5, 2.5, 4.5, 3.4e+5)</td>
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<tr>
<td>(0.5, 0.5, 0.5, 5.0e+4)</td>
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<td>338.1</td>
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<td>(3.5, -3.5, 2.5, 1.0e+4)</td>
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<tr>
<td>(-2.5, 2.5, 4.5, 1.0e+4)</td>
<td>(-2.5, 2.5, 4.5, 3.3e+3)</td>
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<tr>
<td>(0.5, 0.5, 0.5, 5.0e+6)</td>
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<td>6.78e-1</td>
<td>890.8</td>
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<td>(0.5, 0.5, 3.5, 3.7e+5)</td>
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<td>(0.5, 0.5, 0.5, 1.6e+4)</td>
<td>6.68e-1</td>
<td>722.4</td>
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<td>(-2.5, 2.5, 4.5, 3.3e+3)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** MC recovery examples with iterPOMP algorithm.
Conclusions and future work

- Sparsity improves location accuracy
- $\ell^0$ regularization is more robust than $\ell^1$ regularization.
- $\ell^1$ fails due to high correlation (ill-condition) of forward system matrix: higher order minimization algorithm for $\ell^1$ poisson model?
- Poisson OMP matching pursuit is efficient for small number of sources reconstruction problem. Parallelizable and need less parameters compared to $\ell^1$ minimization.
- Poisson OMP is a greedy algorithm. Need to improve the efficiency by a faster proximal solution
- Theoretical study on $\ell^1$ and $\ell^0$ reconstruction model for BLT problem.
- Test on real data, incomplete data with more complicated 3D body.
- Extension to total variation or framelets regularization together with better FEM discretization and more complicated 3D shapes.