The Laplace-Beltrami Operator in a Level Set Framework and Its Applications

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Laplace-Beltrami Operator

Laplace-Beltrami: surface version of the Laplace operator.

\( \Gamma \): a closed surface embedded in \( \mathbb{R}^{d+1} \) with metric \( g \).

\( \forall \ p \in \Gamma, \ \exists \ \text{a neighborhood} \ U \ \text{isomorphic to} \ V \in \mathbb{R}^d. \)

Laplace-Beltrami operator \( \Delta_{\Gamma} : C^\infty(\Gamma) \rightarrow C^\infty(\Gamma) \)

\[
\Delta_{\Gamma} u = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\sqrt{|g|} \sum_{j=1}^{d} g^{ij} \frac{\partial u}{\partial x_j}) \tag{1}
\]

where \( x = (x_1, \cdots, x_d) \in V, \ u \in C^\infty(\Gamma), \ (g^{ij}) \) is the inverse matrix of \( (g_{ij}) \) and \( |g| = \det(g_{ij}). \)
Surfaces or interfaces can be represented as the zero level set of a function $\phi \in C(\mathbb{R}^{d+1})$.

$$\Gamma = \{ x \in \mathbb{R}^{d+1} : \phi(x) = 0 \}.$$ 

Signed Distance Function (SDF):

$$\phi(x) = \begin{cases} 
- \min_{y \in \Gamma} ||x - y|| & \text{if } x \text{ is inside } \Gamma, \\
\min_{y \in \Gamma} ||x - y|| & \text{if } x \text{ is outside } \Gamma. 
\end{cases}$$

Property of SDF: $|\nabla \Phi| = 1$. 

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Geometric Features

Outer unit normal vector of the surface $\Gamma$:

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}.$$ 

Mean curvature of the surface $\Gamma$:

$$h = \text{div} \vec{n} = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}.$$ 

For signed distance function:

$$\vec{n} = \nabla \phi,$$

$$h = \nabla \cdot \nabla \phi = \Delta \phi.$$
Level Set Representation of Operators

Surface gradient operator:

\[ \nabla_{\Gamma} u = \nabla u - (\nabla u \cdot \vec{n}) \vec{n} = \nabla u - \frac{\nabla u \cdot \nabla \phi}{|\nabla \phi|^2} \nabla \phi. \quad (2) \]

Surface divergence operator: \( \text{div}_{\Gamma} \vec{v} = \nabla_{\Gamma} \cdot \vec{v} \).

Laplace-Beltrami operator:

\[ \Delta_{\Gamma} u = \nabla_{\Gamma} \cdot \nabla_{\Gamma} u \]
\[ = \Delta u - \nabla \cdot \left( \frac{\nabla u \cdot \nabla \phi}{|\nabla \phi|^2} \nabla \phi \right). \quad (3) \]
Interface Motion Equations

For $\forall \, x \in \Gamma$, let $v_n(x, t) = \frac{\partial x}{\partial t} \cdot \vec{n}$ be the normal speed.

Evolution equation under a level set framework:

$$\phi_t + v_n |\nabla \phi| = 0.$$  

Examples:
- $v_n = -h$: motion by mean curvature.
- $v_n = \Delta_{\Gamma} h$: motion by surface Laplacian of mean curvature.
- $v_n = \Delta_{\Gamma} G(h)$: general surface diffusion.
Interface Motion Equations

Volume conservation:

\[
\frac{d}{dt} |\Omega(t)| = \int_{\Gamma(t)} v dA = - \int_{\Gamma(t)} x_t \cdot \vec{n} dA \\
= - \int_{\Gamma(t)} \text{div}_\Gamma (g(h) \nabla_\Gamma h) dA = - \int_{\Gamma(t)} g(h) \nabla_\Gamma h \cdot \nabla_\Gamma 1 dA = 0.
\]

Area shrinkage:

\[
\frac{d}{dt} |\Gamma(t)| = - \int_{\Gamma(t)} vh dA = - \int_{\Gamma(t)} hx_t \cdot \vec{n} dA \\
= \int_{\Gamma(t)} h \text{div}_\Gamma (g(h) \nabla_\Gamma h) h dA = - \int_{\Gamma(t)} g(h) \nabla_\Gamma h \cdot \nabla_\Gamma h dA \leq 0.
\]
A Splitting Scheme

General surface diffusion: \( v_n = \Delta_{\Gamma} G(h) \).

Evolution equation under a level set framework:

\[
\phi_t + |\nabla \phi| \Delta_{\Gamma} G(h) = 0,
\]

where \( h = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} \).

Numerical difficulties: fourth order and nonlinear.

- Explicit schemes: very small time step, \( dt \sim dx^4 \).
- Implicit schemes: solving a complex system.
A semi-implicit scheme can be obtained by adding bilaplacian stabilization terms:

\[ \phi_t + \eta \Delta^2 \phi = \eta \Delta^2 \phi - |\nabla \phi| \Delta_T G(h). \]

Take the left hand bilaplacian implicit and take right hand explicit:

\[
\frac{\phi^{n+1} - \phi^n}{dt} + \eta \Delta^2 \phi^{n+1} = \eta \Delta^2 \phi^n - |\nabla \phi^n| \Delta_T G(h^n),
\]

\[
(1 + dt\eta \Delta^2)(\phi^{n+1} - \phi^n) = -dt|\nabla \phi^n| \Delta_T G(h^n).
\]
Numerical Discretization

Discretization: use $|\nabla \phi|_\delta = \sqrt{\phi_x^2 + \phi_y^2 + \delta^2}$ to avoid division by zero.

\[
\vec{n}_\delta = \frac{\nabla \phi}{|\nabla \phi|_\delta}, \quad h_\delta = \text{div} \frac{\nabla \phi}{|\nabla \phi|_\delta} = \frac{\nabla \phi}{|\nabla \phi|_\delta} - \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{|\nabla \phi|_\delta^3}.
\]

Modified equation and corresponding numerical scheme:

\[
\phi_t = -|\nabla \phi|_\delta \Delta \Gamma G(h^k_\delta), \quad (4)
\]

\[
\frac{\Phi^{k+1} - \Phi^k}{dt} + \eta \Delta^2 \Phi^{k+1} = \eta \Delta^2 \Phi^k - |\nabla \Phi^k|_\delta \Delta \Gamma G(h^k_\delta). \quad (5)
\]
Theorem 1. Let $\phi$ be the exact solution of (4) and $\phi^k = \phi(kdt)$ be the exact solution at time $kdt$ for a time step $dt > 0$ and $k \in \mathbb{N}$. Let $\Phi^k$ be the $k$th iterate of (5). Assume that there exits a constant $L$ such that $|G'(s)| \leq L$, $|G''(s)| \leq L$, and the discrete solution exists up to time $T$, then we have the following statements:

(i) Under the assumption that $\|\phi_{tt}\|_{-1}$, $\|\nabla \Delta \phi_t\|_2$, $\|\nabla \phi\|_{\infty}$ and $\|\phi_t\|_{-1}$ are bounded, the numerical scheme (5) is consistent with the modified continuous equation (4) and first order in time.
(ii) Let further \( e^k = \phi^k - \Phi^k \) be the discretization error. If

\[
\| \partial^\alpha \phi^k \|_\infty \leq K, \quad \| \nabla \Phi^k \|_\infty \leq K,
\]

for a constant \( K > 0 \) and all \( |\alpha| \leq 3, kdt \leq T \), then the error \( e^k \) converges to zero with first order in time.

Remark:

- Spatial discretization may impose additional restrictions on time step \( dt \). Numerical tests show \( dt \sim dx^2 \).
- Reinitialization is required if derivatives of \( \phi \) are large.
Numerical Results

Motion by surface Laplacian of mean curvature: \( G(x) = x \).

Figure: Merging of a circle and an ellipse.
Numerical Results

Figure: Splitting of a 3D dumbbell.
Active Contour Model - Snake

The classical snake active contour models use an edge-detector, depending on the gradient of the image $u_0$, to stop the evolving curve on the boundary of the desired object.

The Snake model minimizes the following energy:

$$J_1(C) = \alpha \int_0^1 |C'(s)|^2 ds + \beta \int_0^1 |C''(s)| ds - \lambda \int_0^1 |\nabla u_0(C(s))| ds.$$
Chan-Vese Segmentation Model

Level set representation: \( C = \{(x, y) : \phi(x, y) = 0\} \). \( \phi > 0 \) inside the contour and \( \phi < 0 \) outside the contour.

The Chan-Vese model minimizes the following energy:

\[
F(\phi, c_1, c_2) = \mu \int_{\Omega} \delta(\phi) |\nabla \phi| \, dx\,dy + \nu \int_{\Omega} H(\phi) \, dx\,dy \\
+ \lambda_1 \int_{\Omega} (u_0 - c_1)^2 H(\phi) \, dx\,dy + \lambda_2 \int_{\Omega} (u_0 - c_2)^2 (1 - H(\phi)) \, dx\,dy.
\]

\( H \) is the heaviside function:

\[
H(x) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0.
\end{cases}
\]
ʃ is the one dimensional Dirac measure: ʃ(x) = d/dx H(x).

The gradient descent equation is

\[ c_1 = \frac{\int_{\Omega} u_0 H(\phi) dx dy}{\int_{\Omega} H(\phi) dx dy}, \quad c_2 = \frac{\int_{\Omega} u_0 (1 - H(\phi)) dx dy}{\int_{\Omega} (1 - H(\phi)) dx dy}, \]

\[ \phi_t = \delta(\phi) \left[ \mu \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} - \nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \right]. \quad (6) \]
Low Curvature Image Simplifier

Fourth order *Low Curvature Image Simplifier* (LCIS) PDE

\[ u_t + \text{div}(g(\Delta u)\nabla \Delta u) = 0. \]  \hspace{1cm} (7)

\( g \geq 0 \): a weight function satisfying \( g(0) = 1 \) and \( g(\infty) = 0 \).

Mass conservation:

\[ \frac{d}{dt} \int u dx = \int u_t dx = - \int \text{div}(g(\Delta u)\nabla \Delta u) dx = 0. \]

\( H_1 \) energy decay:

\[ \frac{d}{dt} \int |\nabla u|^2 dx = \int 2\nabla u \nabla u_t dx = - \int 2\Delta uu_t dx \\
= \int 2\Delta u \text{div}(g(\Delta u)\nabla \Delta u) dx = - \int 2g(\Delta u)(\nabla \Delta u)^2 dx \leq 0. \]
Low Curvature Image Simplifier

LCIS denoising model:

\[ u_t + \text{div}(g(\Delta u)\nabla \Delta u) = \lambda(f - u). \]  \hspace{1cm} (8)
A Geometric Variant of LCIS

A geometric variant of LCIS PDE:

\[ x_t - \text{div}_\Gamma (g(h) \nabla_\Gamma \Delta_\Gamma x) = 0. \] (9)

Note that \( \Delta_\Gamma x = h\vec{n} \). Equation (9) is very similar to the surface diffusion equation:

\[ x_t - \text{div}_\Gamma (g(h) \nabla_\Gamma h)\vec{n} = 0. \] (10)

Level set representation of equation (10):

\[ \phi_t = -v|\nabla \phi| = -|\nabla \phi| \text{div}_\Gamma (g(h) \nabla_\Gamma h). \] (11)
Geometric LCIS Evolution

The evolution of a curve under equation (11).

Figure: Corner formation in early stages.
Chan-Vese Segmentation Model with Corners

The regularization term, or the curve length term can avoid the noisy or undesired pieces, it rounds off the corners at the same time.

We can add equation (11) in to the Chan-Vese model. The new equation becomes:

\[
\phi_t = \delta(\phi) [\mu \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} - \nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2] \\
- \alpha |\nabla \phi| \text{div}_\Gamma (g(h) \nabla_\Gamma h).
\]

(12)
A Simple shape

(a) segmentation without corner term; (b) segmentation with corner;
A Building

(a) segmentation without corner term; (b) segmentation with corner;
A Walmart

(a) segmentation without corner term; (b) segmentation with corner;

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Laplace-Beltrami Eigen Structure

Eigen structure problem:

\[ \Delta_{\Gamma} u = -\lambda u \quad \text{on } \Gamma. \]  \hspace{1cm} (13)

- Brandman: embed the shape in 3D domain and approximate the LB spectrum with the spectrum of a 3D elliptic operator. Disadvantage: computational cost and accuracy.
Triangular Mesh Representation

Example: a sphere (left) and the triangular mesh (right).
Narrow Band and Parametrization

Narrow Band with width $\delta > 0$: $\Gamma_\delta = \{ x : |\phi(x)| < \delta \}$.

New parametrization of the narrow band:

- For any $p \in \Gamma$, we choose an isothermal coordinate $(x_1, x_2)$ around $p$ on the surface, i.e., $\langle \partial_{x_i}, \partial_{x_j} \rangle = \delta_{ij}$, where $\partial_{x_i}$ denotes the tangent vector in $x_i$ direction. The normal direction is $\vec{n} = \partial_{x_1} \times \partial_{x_j}$.

- For any $q \in \Gamma_\delta$, let $p = \text{Proj}_{\Gamma} q$, then $q = p + \phi(q)\vec{n}$. $q$ can be parameterized as $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, where $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = \phi(q)$.

- This new coordinate is unique iff $\delta < \frac{1}{\max_p \kappa(p)}$. 
Laplace-Beltrami and Laplace Operator

Under this parametrization, for any $p \in \Gamma$, the Laplace-Beltrami operator is

$$\Delta_\Gamma u = \left( \frac{\partial^2}{\partial^2 x_1} + \frac{\partial^2}{\partial^2 x_2} \right) u. \quad (14)$$

The Laplace operator is

$$\Delta u = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{3} \frac{\partial}{\partial \tilde{x}_i} \left( \sqrt{|g|} \ g^{ij} \frac{\partial}{\partial \tilde{x}_j} u \right),$$

where $g = (g_{ij})$ is the Euclidean metric of the narrow band expressed in the parametrization $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, $g_{ij} = \langle \partial_{\tilde{x}_i}, \partial_{\tilde{x}_j} \rangle$, $(g^{ij})$ is the inverse matrix of $g$ and $|g| = \det(g_{ij})$. 
Under the parametrization, we have

\[ \partial \tilde{x}_1 = \partial x_1 + \tilde{x}_3 \vec{n}_{x_1}, \quad \partial \tilde{x}_2 = \partial x_2 + \tilde{x}_3 \vec{n}_{x_2}, \quad \partial \tilde{x}_3 = \vec{n}. \]

Thus we have

\[
\begin{align*}
\langle \partial \tilde{x}_1, \partial \tilde{x}_1 \rangle &= 1 + 2\tilde{x}_3 \langle \partial x_1, \vec{n}_{x_1} \rangle + \tilde{x}_3^2 \| \vec{n}_{x_1} \|^2, \\
\langle \partial \tilde{x}_1, \partial \tilde{x}_2 \rangle &= \tilde{x}_3 (\langle \partial x_1, \vec{n}_{x_2} \rangle + \langle \partial x_2, \vec{n}_{x_1} \rangle) + \tilde{x}_3^2 \langle \vec{n}_{x_1}, \vec{n}_{x_2} \rangle, \\
\langle \partial \tilde{x}_2, \partial \tilde{x}_2 \rangle &= 1 + 2\tilde{x}_3 \langle \partial x_2, \vec{n}_{x_2} \rangle + \tilde{x}_3^2 \| \vec{n}_{x_2} \|^2, \\
\langle \partial \tilde{x}_1, \partial \tilde{x}_3 \rangle &= \langle \partial \tilde{x}_2, \partial \tilde{x}_3 \rangle = 0, \\
\langle \partial \tilde{x}_3, \partial \tilde{x}_3 \rangle &= 1.
\end{align*}
\]
The second fundamental form is

\[
\begin{aligned}
\vec{n}_{x_1} &= -\frac{L}{E} \partial_{x_1} - \frac{M}{G} \partial_{x_2}, \\
\vec{n}_{x_2} &= -\frac{M}{E} \partial_{x_1} - \frac{N}{G} \partial_{x_2},
\end{aligned}
\]

where \( E = \langle \partial_{x_1}, \partial_{x_1} \rangle, \ G = \langle \partial_{x_2}, \partial_{x_2} \rangle, \ L = \langle \partial_{x_1} \partial_{x_1}, \vec{n} \rangle, \ M = \langle \partial_{x_1} \partial_{x_2}, \vec{n} \rangle, \ N = \langle \partial_{x_2} \partial_{x_2}, \vec{n} \rangle. \)

Under the isothermal parameterization, we have \( E = 1, \ G = 1. \)
Laplace Operator Under New Parametrization

The Euclidean metric of the narrow band under the new parametrization is

\[
\begin{align*}
\tilde{g} = (\tilde{g}_{ij}) &= \text{Id} + \tilde{x}_3 \begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix} \\
A &= \begin{pmatrix}
-2L + \tilde{x}_3(L^2 + M^2) & -2M + \tilde{x}_3(LM + MN) \\
-2M + \tilde{x}_3(LM + MN) & -2N + \tilde{x}_3(M^2 + N^2)
\end{pmatrix}.
\end{align*}
\]
Consequently we have

\[
\sqrt{|\widetilde{g}|} = det(\widetilde{g}_{ij}) = 1 + \frac{1}{2} \tilde{x}_3 (A_{11} + A_{22}) + O(\tilde{x}_3^2),
\]

\[
\widetilde{g}^{-1} = \begin{pmatrix}
Id - \tilde{x}_3 A + O(\tilde{x}_3^2) & 0 \\
0 & 1
\end{pmatrix}.
\]

We obtain

\[
\Delta u = \left( \frac{\partial^2}{\partial^2 \tilde{x}_1} + \frac{\partial^2}{\partial^2 \tilde{x}_2} \right) u + \frac{\partial^2}{\partial^2 \tilde{x}_3} u - 2\kappa \frac{\partial}{\partial \tilde{x}_3} u + O(\tilde{x}_3). \tag{15}
\]
We define
\[ \Delta_{12} u = \left( \frac{\partial^2}{\partial^2 \tilde{x}_1} + \frac{\partial^2}{\partial^2 \tilde{x}_2} \right) u, \]
\[ \tilde{\Delta} u = \Delta_{12} u + \frac{\partial^2}{\partial^2 \tilde{x}_3} u. \]

Consider the following Neumann boundary condition problem:
\[
\begin{cases}
\tilde{\Delta} u = \Delta_{12} u + \frac{\partial^2}{\partial^2 \tilde{x}_3} u = -\lambda u & \text{in } \Gamma_\delta, \\
\frac{\partial u}{\partial \tilde{x}_3} = 0 & \text{for } \tilde{x}_3 = \pm \delta.
\end{cases}
\]
**Theorem:** The eigen structure for (16) has the following decomposition

\[
\begin{align*}
  u_{m,n} &= u_{m}^{G}(\tilde{x}_{1}, \tilde{x}_{2})u_{n}^{I}(\tilde{x}_{3}), \\
  \lambda_{m,n} &= \lambda_{m}^{G} + \lambda_{n}^{I},
\end{align*}
\]

(17)

where \(\lambda_{m}^{G}\) and \(u_{m}^{G}(\tilde{x}_{1}, \tilde{x}_{2}) = u_{m}^{G}(x_{1}, x_{2})\) are the eigenvalues and eigenfunctions for (13), while \(\lambda_{n}^{I}\) and \(u_{n}^{I}\) are the eigenvalues and eigenfunctions for the one dimensional problem

\[
\begin{align*}
  \frac{d^2}{d^2 t} u &= -\lambda u, \quad t \in (-\delta, \delta) \\
  \frac{d}{dt} u &= 0, \quad t = \pm \delta.
\end{align*}
\]

(18)
Eigen Structure Decomposition

(18) has following eigenvalues and eigenfunctions

\[ \lambda_0^I = 0, \quad u_0^I = \text{const}, \]
\[ \lambda_n^I = \frac{(2n - 1)^2 \pi^2}{4 \delta^2}, \quad u_n^I = \sin((n - 1/2)\pi/\delta), \quad n = 1, 2, \ldots \]

Since \( \Delta \Gamma \) is elliptic on \( \Gamma \), we can order the eigenvalues

\[ 0 = \lambda_1^\Gamma \leq \lambda_2^\Gamma \leq \cdots \leq \lambda_m^\Gamma \leq \cdots . \]

For \( \delta \) small enough, the eigenvalues of \( \tilde{\Delta} \) can be ordered as

\[ \lambda_1^\Gamma \leq \lambda_2^\Gamma \leq \cdots \leq \lambda_m^\Gamma \leq \lambda_1^\Gamma + \lambda_1^\Gamma \leq \lambda_1^\Gamma + \lambda_2^\Gamma \leq \cdots . \]

The corresponding eigenfunctions of \( \tilde{\Delta} \)

\[ u_{k,0} = u_k^\Gamma \cdot \text{const}, \quad k = 1, \cdots , m. \]
Approximation

For $\delta$ small enough,

$$
\Delta u_{k,0} = \tilde{\Delta} u_{k,0} + O(\tilde{x}_3) = \lambda_{k,0} u_{k,0} + O(\delta). \quad (19)
$$

If $\delta \sim dx$, then $O(\delta) = O(dx)$. $\Delta$ is a very good approximation to $\tilde{\Delta}$ for the eigen structure problem.

Solve the following problem to approximate the Laplace-Beltrami eigen structure:

$$
\begin{cases}
\Delta u = -\lambda u & \text{in } \Gamma_\delta, \\
\frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial \Gamma_\delta.
\end{cases} \quad (20)
$$
Weak Formulation

Definition: $u$ is weak solution to

$$\Delta u = -\lambda u \quad \text{in } \Gamma_\delta, \quad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \partial \Gamma_\delta$$

if and only if

$$\left( \nabla u, \nabla v \right) = -\lambda (u, v) \quad \text{for any } v \in H_1(\Gamma_\delta).$$

Implementation with finite element is easy.

Choose cubes instead of tetrahedra as basic elements and tri-linear basis functions.
Weak Formulation

Figure: The cubic elements forming the narrow band of a sphere.
Numerical Results

Numerical experiments on anatomical shapes: caudate, putamen and hippocampus.

Comparison: narrow band method vs. mesh based method.

For comparison of eigenfunctions, we define the correlation coefficient of two functions:

\[
\text{corr}(u, v) = \frac{\int_{\Gamma} uv \, d\Gamma}{\sqrt{\int_{\Gamma} u^2 \, d\Gamma} \sqrt{\int_{\Gamma} v^2 \, d\Gamma}}. \tag{21}
\]

**Attention:** eigenvalues with multiplicity greater than 1 may occur!
Comparison: Anatomical Shapes

Figure: The shapes used in our experiments. Top: Caudate. Middle: Putamen. Bottom: Hippocampus.
Narrow Band vs. Mesh: Eigenvalues

Figure: The eigenvalues computed from the mesh and narrow band representation for the shapes.
Narrow Band vs. Mesh: Eigenfunctions

Figure: The 2nd, 3rd, 4th eigenfunctions of a caudate surface. Top row: results from the mesh representation. Bottom row: results from the narrow band representation.
Figure: The 2nd, 3rd, 4th eigenfunctions of a putamen surface. Top row: results from the mesh representation. Bottom row: results from the narrow band representation.
Figure: The 2nd, 3rd, 4th eigenfunctions of a hippocampal surface. Top row: results from the mesh representation. Bottom row: results from the narrow band representation.
Narrow Band vs. Mesh: Correlation Coefficients

Figure: The correlation coefficient between the eigenfunctions computed with mesh based and narrow band approaches.
Narrow Band vs. Mesh: Adjusted Correlation Coefficients

Small correlation coefficients occur!

Adjustment: project narrow band eigenfunctions onto the span of mesh based eigenfunctions.

Figure: The adjusted correlation coefficient between the eigenfunction computed with the narrow band approach and the mesh based approach.
Thank you!