A Narrow-Band Approach for Approximating the Laplace-Beltrami Spectrum of 3D Shapes

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Abstract. The spectrum of the Laplace-Beltrami operator is a useful tool for various shape analysis problems as it provides an intrinsic, Fourier-like characterization of 3D shapes. The numerical computation of the Laplace-Beltrami spectrum is typically based on a triangular mesh representation of the 3D surface, which makes its robustness dependent upon the quality of triangulation. In this paper, we propose an alternative approach by approximating the Laplace-Beltrami spectrum with the Laplacian spectrum of a narrow band composed of regular cubes with the Neumann boundary condition, thus mesh quality is not an issue in our method. We validate our method by computing the spectrum of real anatomical shapes and comparing with the mesh-based method.

Keywords: Laplace-Beltrami operator, spectrum, narrow band, level set, shape analysis.

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INTRODUCTION

The development of robust and intrinsic characterization of 3D anatomical shapes is a challenging problem in shape analysis, especially medical image analysis [1, 2, 3, 4, 5]. Recently there have been increasing interests in using the spectrum of the Laplace-Beltrami operator to study 3D medical shapes and various applications as in [6, 7, 8]. However, the numerical computation of the spectrum can be difficult.

To compute the spectrum of surfaces, numerical algorithms based on their triangular mesh representation were adopted in previous works, which makes their robustness dependent upon the quality of triangulation as discussed in [7]. In order to overcome this limitation, Brandman proposed to embed the shape in a rectangular domain and approximate the Laplace-Beltrami spectrum of the surface with the spectrum of a new elliptic operator defined over this Euclidean domain [9], consequently independent of the mesh quality. While this method is theoretically very elegant, the computational accuracy and efficiency need to be improved in 3D case.

Motivated by the success of narrow bands in the numerical implementation of the level-set method[10, 11], we propose an alternative approach of computing surface spectrum. In addition, geometric quantities of the surface such as curvature and intrinsic gradients can be computed from the signed distance function defined on the narrow-band[12]. Based on this observation, we propose in this paper to use the Laplacian spectrum of a narrow band surrounding the surface to approximate the Laplace-Beltrami spectrum of the surface. Under the Neumann boundary condition, we show that very accurate results can be obtained. Our method is based on the narrow band, thus more efficient than [9].

The rest of the paper is organized as follows. In section , we derive the mathematical foundation of our approximation. In section we develop the numerical algorithm to compute the spectrum using finite element methods. Experimental results are then presented in section on anatomical structures: the caudate nucleus and putamen. Finally conclusions are made in section .

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**MATHEMATICAL FORMULATION**

In this section, we derive the relation between the Laplace-Beltrami operator on a surface and the Laplace operator of a narrow band surrounding the surface. Then we develop the narrow-band approximation of the Laplace-Beltrami spectrum of the surface.

Let \((\mathcal{M}, g)\) be a Riemannian surface in \(\mathbb{R}^3\) [13]. For any point \(p \in \mathcal{M}\), it has a neighborhood \(U \subset \mathcal{M}\) that is diffeomorphic to a subset \(V \subset \mathbb{R}^2\). With this parameterization, the Laplace-Beltrami operator \(\Delta_{\mathcal{M}} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\) on \(\mathcal{M}\) can be expressed as:

\[
\Delta_{\mathcal{M}} u = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} \sum_{j=1}^{3} g^{ij} \frac{\partial u}{\partial x_j} \right)
\]  

where \((x_1, x_2) \in V\), and \(u \in C^\infty(\mathcal{M})\) is a function defined on \(\mathcal{M}\), \((g^{ij})\) is the inverse matrix of \((g_{ij})\) and \(|g|=\det(g_{ij})\).

The spectrum of the Laplace-Beltrami operator [14] is defined with the following equation:

\[
\Delta_{\mathcal{M}} u = -\lambda u
\]

where \(\lambda\) is the eigenvalue and \(u \in C^\infty(\mathcal{M})\) is the eigenfunction. Because the operator \(\Delta_{\mathcal{M}}\) is self-adjoint and elliptic, its spectrum is discrete, so we can order the eigenvalues as \(0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\). For the eigenvalue \(\lambda_m\), its eigenfunction is denoted as \(u_m\).

We assume the surface \(\mathcal{M}\) is closed and simple, so we can define a signed distance function (SDF) as:

\[
\phi(x) = \begin{cases}
-\min_{y \in \mathcal{M}} |x-y| & \text{if } x \text{ is inside } \mathcal{M}, \\
\min_{y \in \mathcal{M}} |x-y| & \text{if } x \text{ is outside } \mathcal{M}.
\end{cases}
\]  

Using this SDF, we define a narrow band surrounding the surface \(\mathcal{M}\) as \(\Sigma_\delta = \{x \in \mathbb{R}^3 : |\phi(x)| < \delta\}\).

We next derive the relation between the Laplace operator \(\Delta : C^\infty(\Sigma_\delta) \to C^\infty(\Sigma_\delta)\) on this narrow band and the Laplace-Beltrami operator \(\Delta_{\mathcal{M}}\) of the surface. For any point \(p \in \mathcal{M}\), we can choose an isothermal coordinate \((x_1, x_2)\) in the neighborhood \(U\) on the surface, i.e., \((\partial_{x_1}, \partial_{x_2}) = \delta_{ij}\), where \(\delta_{ij}\) denotes the tangent vector in the \(i\) direction under this parameterization. At this point, the normal direction pointing outward can be expressed as \(\vec{n} = \partial_{x_1} \times \partial_{x_2}\). For each point \(q\) in the narrow band, we project it onto \(\mathcal{M}\) and find its nearest point \(p\) as:

\[
q = p + \phi(q) \vec{n}.
\]  

Thus we can parameterize \(q\) as \((\vec{x}_1, \vec{x}_2, \vec{x}_3)\) where \(\vec{x}_1 = x_1, \vec{x}_2 = x_2, \vec{x}_3 = \phi(q)\). Under this parameterization, the Laplace operator at the point \(q\) is:

\[
\Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u + \frac{\partial^2}{\partial x_3^2} u - 2\kappa \frac{\partial u}{\partial x_3} + O(\vec{x}_3)
\]

where \(\kappa\) is the mean curvature at the point \(p \in \mathcal{M}\).

Let \(\Delta_{12} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\) and \(\Delta = \Delta_{12} + \frac{\partial^2}{\partial x_3^2}\). To see the relation between the eigen-structure of \(\Delta_{\mathcal{M}}\) and \(\Delta_{\mathcal{M}}\), we consider the following Neumann boundary problem:

\[
\begin{cases}
\tilde{\Delta} u = \Delta_{12} u + \frac{\partial^2}{\partial x_3^2} u = -\lambda u & \text{in } \Sigma_\delta, \\
\frac{\partial u}{\partial x_3} = 0 & \text{for } \vec{x}_3 = \pm \delta.
\end{cases}
\]  

Let \(\lambda_{m}^I\) and \(u_{m}^I(\vec{x}_3)\) be the eigenvalues and eigenfunctions of the one dimensional problem on \(I = [-\delta, \delta]\):

\[
\frac{d^2 u}{dt^2} = -\lambda^I u, \quad t \in (-\delta, \delta)
\]

with boundary condition \(\frac{du}{dt} = 0\) for \(t = -\delta, \delta\). Because \(\tilde{x}_1 = x_1, \tilde{x}_2 = x_2\) and \(\Delta_{12} = \Delta_{\mathcal{M}}\) at \(\tilde{x}_3 = 0\), we have \(\Delta_{12} u_m^I(\tilde{x}_1, \tilde{x}_2) = \Delta_{\mathcal{M}} u_m^I(\tilde{x}_1, \tilde{x}_2) = -\lambda_m^I u_m^I(\tilde{x}_1, \tilde{x}_2)\). Using separation of variables, we obtain the eigenvalues and eigenfunctions of \(\Delta\) on the narrow band as:

\[
\lambda_{m,n} = \lambda_m^I + \lambda_n^I, \quad u_{m,n} = u_m^I u_n^I, \quad m, n = 1, 2, \ldots.
\]
For the one dimensional problem in (7), its first eigenvalue is \( \lambda_1 = 0 \) with the eigenfunction \( u_1 = \text{const} \), and its second eigenvalue is \( \lambda_2 = \frac{\pi^2}{4a^2} \) with the corresponding eigenfunction \( u_2 = \sin \frac{\pi}{2a}t \). Therefore, if we want to extract the first \( m \) eigenvalues and eigenfunctions of \( \Delta \) from the eigen-structure of \( \tilde{\Delta} \), we just choose \( \delta \) small enough such that \( \lambda_2^\text{corr} > \lambda_m \) and then we can order the eigenvalues of \( \Delta \) as follows

\[
\lambda_1^\text{corr}, \lambda_2^\text{corr}, \ldots, \lambda_m^\text{corr}, \ldots, \lambda_2^\text{corr} + \lambda_1^\text{corr}, \lambda_2^\text{corr} + \lambda_2^\text{corr}, \ldots
\]

(9)

The corresponding eigenfunctions of the first \( m \) eigenvalues are \( u_{m,1}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = u_1^1(\hat{x}_1)u_m^\text{corr}(\hat{x}_1, \hat{x}_2) \).

Now we apply the Laplace operator \( \Delta \) to the eigenfunctions \( u_{m,1} \). Note that \( u_{m,1} = c \cdot u_m^\text{corr} \) is constant in the normal direction, we have \( \frac{\partial}{\partial x_3} u_{m,1} = 0 \). Without loss of generality, we can take the constant \( c = 1 \), then

\[
\Delta u_{m,1} = \Delta u_m^\text{corr} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u_m^\text{corr} + \frac{\partial^2}{\partial x_3^2} u_m^\text{corr} - 2k \frac{\partial}{\partial x_3} u_m^\text{corr} + O(\tilde{x}_3) = -\lambda_m u_{m,1} + O(\tilde{x}_3),
\]

(10)

which means that \( u_{m,1} \) are almost the eigenfunctions of \( \Delta \) with eigenvalues \( \lambda_m^\text{corr} \).

This provides us a natural approximation approach of computing the eigenvalues of \( \Delta \) from the eigenvalues of \( \Delta \). We propose here to solve the following eigenvalue problem of the Laplace operator \( \Delta \) over the narrow band:

\[
\begin{cases}
    \Delta u = -\lambda u & \text{in } \Sigma_\delta \\
    \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Sigma_\delta.
\end{cases}
\]

(11)

After we solve (11), by \( u_{m,1} = \text{const} \cdot u_m^\text{corr} \), we can obtain the eigenfunctions of \( \Delta \) easily.

**NUMERICAL IMPLEMENTATION**

In this section, we develop the numerical algorithm to solve the eigenvalue problem in (11). We convert the problem in (11) into a weak form and use the classic finite element method to obtain the eigen-structure.

Let \((\cdot, \cdot)\) be the inner product under the \( L_2 \) norm over the narrow band \( \Sigma_\delta \), i.e., \( (u, v) = \int_{\Sigma_\delta} uv dx, \forall u, v \in L_2(\Sigma_\delta) \). We define the Sobolev space \( H^1(\Sigma_\delta) \) as \( H^1(\Sigma_\delta) = \{ u \in L_2(\Sigma_\delta): \nabla u \in L_2(\Sigma_\delta) \} \).

For any test function \( v \in H^1(\Sigma_\delta) \), using the Neumann boundary condition, we have

\[
(-\Delta u, v) = \int_{\Sigma_\delta} \Delta u \cdot v dx = \int_{\partial \Sigma_\delta} \frac{\partial u}{\partial n} v dx + \int_{\Sigma_\delta} (\nabla u, \nabla v) dx = \int_{\Sigma_\delta} (\nabla u, \nabla v) dx = (\nabla u, \nabla v),
\]

(12)

Following the method in [15], we have the weak formulation of the eigen-structure problem over the narrow band as summarized by the following theorem, then we can apply the classic finite element method to solve it.

**Theorem 1** \( u \in H^1(\Sigma_\delta) \) is a weak solution of (11) if and only if

\[
(\nabla u, \nabla v) = \lambda (u, v) \quad \text{for all } v \in H^1(\Sigma_\delta).
\]

(13)

**EXPERIMENTAL RESULTS**

In this section, we present experimental results to illustrate our narrow-band approach for computing the Laplace-Beltrami spectrum of 3D shapes. We demonstrate our algorithm on the caudate nucleus in the brain. We show the effectiveness of our algorithm by comparing with conventional approaches using triangular meshes.

We computed the first 25 eigenvalues and corresponding eigenfunctions. We denote \( \tilde{\lambda}_i \) and \( \tilde{u}_i \) as the eigenvalues and eigenfunctions computed from the mesh representation, and the eigenvalues and eigenfunctions computed from the narrow-band approach as \( \lambda_i \) and \( \tilde{u}_i \), respectively. Following previous section, the eigenfunction \( \tilde{u}_i \) obtained from the narrow-band approach corresponds to the eigenfunction \( u_i^\text{corr} \) multiplied by a constant, thus we use the correlation coefficient to measure the difference between \( \tilde{u}_i \) and \( \tilde{u}_i \) as defined by:

\[
\text{corr}(\tilde{u}_i, \tilde{u}_i) = \frac{\int_{\Delta} \tilde{u}_i \tilde{u}_i d.\mathcal{M}}{\sqrt{\int_{\Delta} \tilde{u}_i^2 d.\mathcal{M}} \sqrt{\int_{\Delta} \tilde{u}_i^2 d.\mathcal{M}}}
\]

(14)

We can see the narrow-band approach precisely follow the mesh-based results.
CONCLUSIONS

In this paper we proposed a narrow-band approach for the numerical computation of Laplace-Beltrami spectrum on closed surfaces. Our method achieves comparable results to the mesh-based approach for the analysis of 3D shapes, yet is independent of mesh quality because we discretize the narrow band with uniform cubic elements. Experimental results on three subcortical structures in the brain have been presented to demonstrate the usefulness of our method.

REFERENCES