An improved nonlocal means filter for image processing

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Abstract

Recently the nonlocal means filter introduced by Buades-Coll-Morel plays an important role in image processing. By introducing nonlocal operators, Gilboa-Osher proposed various nonlocal models, which improved the nonlocal means filter. In recent papers with Jost and Wang [1], [2] we presented new nonlocal operators and various new nonlocal models for image processing, among which a nonlocal \( H^1 \) model was proposed. Motivated by this new nonlocal \( H^1 \) model we propose in this paper a simpler model for image processing, which is a suitable improvement of the nonlocal means filter. We compare this model with the nonlocal means filter, both theoretically and in experiments.

Keywords: Nonlocal means filter, image denoising, nonlocal \( H^1 \) model, TV model

1 Introduction

Due to a noise \( v \) we may obtain an image \( f = u + v \) from an original image \( u \). In denoising, we want recover the original image \( u \) from the noisy image \( f \). This is an inverse problem, and in general it does not work.

Motivated by the Yaroslavsky filter [3], Buades-Coll-Morel [4] proposed the nonlocal means filter (NLM) for image denoising. The model is

\[
NLM(u)(x) = \frac{1}{c(x)} \int_{\Omega} e^{-d_\sigma(u(x),u(y))/h^2} u(y) \, dy
\]

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where $d_\sigma$ is defined by
\[
d_\sigma(u(x), u(y)) = \int_\Omega G_\sigma(t) |u(x + t) - u(y + t)|^2 dt, \tag{2}
\]
$G_\sigma$ is a Gaussian with standard deviation $\sigma$, and $c(x)$ is defined by
\[
c(x) = \int_\Omega e^{-d_\sigma(u(x), u(y)) / h^2} u(y) dy. \tag{3}
\]
Let $f : \Omega \to \mathbb{R}$ be a noisy image to be denoised. We define
\[
\omega(x, y) = \exp \left( -\frac{d_\sigma(f(x), f(y))}{h^2} \right)
\]
as a weight that measures the similarity of $f$ between the neighborhood of $x$ and that of $y$. Formula (1) is equivalent to
\[
u(x) = N(f) := \frac{1}{\bar{\omega}(x)} \int_\Omega f(y) \omega(x, y) dy, \tag{4}
\]
where $\bar{\omega}$ is defined by
\[
\bar{\omega}(x) = \int_\Omega \omega(x, y) dy.
\]
The resulted image $\nu$ is a denoised image for $f$. It is interesting to see that $\nu$ gives us a very good denoised image in many applications, though the model is very simple. This indicates that the weight $\omega$ captures the feature of images very well. There are other choices of the weight $\omega$. This nonlocal means filter was intensively used in image processing.

Numerically the nonlocal means filter could be written as
\[
N(f)_i = \frac{1}{\sum_j \omega_{ij}} \sum_j \omega_{ij} f_j, \tag{5}
\]
if we denote a digital image by a function $f : [1, N] \times [1, M] \to \mathbb{R}$.

In image processing, the PDE based method plays an important role. In this method one consider the minimizer of suitable functional, which usually satisfies a partial differential equation. Then we use the minimizer as the denoised image. This is one way to study the inverse problem mentioned at the beginning of this paper. The most popular functionals used in image processing are the Dirichlet functional
\[
\int_\Omega |\nabla u|^2(x) dx
\]
and the TV functional
\[
\int |\nabla u|(x) dx.
\]
According to these functionals there are $H^1$ models and TV models for image denoising. Especially the TV model
\[
\int |\nabla u|(x) dx + \lambda \int (u - f)^2(x) dx
\]
proposed by Rudin, Osher and Fatemi [5] is very usefully in image processing. For other PDE based methods, see [6] and [7]. Motivated by the work of Kindermann, Osher and Jones [8], Gilboa and Osher [9, 10, 11] wanted to combine the TV model and the nonlocal means filter into a nonlocal TV model, so that the corresponding \( H^1 \) model is just the regularization of the nonlocal means filter. They realized this idea successfully, by introducing suitable nonlocal operators, which was motivated by the work of Zhou-schölkopf [13, 14]. The nonlocal models they proposed are very useful in image processing. Their models have been used and extended in various problems, see for examples, [15], [16],[17], [18],[19],[20],[21], [22], [23], [24] and [25].

Motivated by the work of Gilboa-Osher and our collaborators’ work in geometric analysis (see for examples, [26]), we proposed new nonlocal operators in [1] and [2]. These new nonlocal operators match the geometric structure of images well by using the weight \( \omega \). With it, we proposed a new nonlocal TV model and a new nonlocal \( H^1 \) model. The new nonlocal TV model was studied in details in [1] and the new nonlocal \( H^1 \) model in [2]. Both models provide good denoising results.

In this paper we focus us on the new nonlocal \( H^1 \) model in [2]. We first present its discrete version. By experiments we found that with only a few iterations in our new nonlocal \( H^1 \) one can already obtain very good results for image denoising. In this paper we only use 2 iterations. This leads to our improved nonlocal means filter (INLM) model

\[
\hat{u} = \text{INLM}(f) := \frac{1}{4} f + \frac{1}{4} N(f) + \frac{1}{2} N^2(f).
\]  

Here \( N \) is defined by (4) and \( N^2(f) = N(N(f)) \) denotes the twice use of the nonlocal means filter \( N \).

The computation of INLM is also very simple. Our experiments show that the model \( \text{INLM} \) is much better than \( N^2 \) and also better than the nonlocal means filter NLM for images with higher textures.

## 2 Discrete version of the new nonlocal \( H^1 \) model

In [1] and [2] we introduced a new nonlocal variational setting, under which we obtained the same nonlocal Dirichlet functional, but a new nonlocal TV model and a corresponding \( H^1 \) model.

In the Section we first present a discrete version of the nonlocal variational setting introduced in [1] and [2].

Now we represent a digital image by a (discrete) function \( u : \Omega \to \mathbb{R} \), where \( \Omega \) is a set \( \{(k, l) \mid 1 \leq k \leq M, 1 \leq l \leq N \} \). Hence \( u \) is a digital image with pixes \( M \times N \). For simplicity, we denote \( u_i + u(i) \) for \( i \in \Omega \). The weight \( \omega : \Omega \times \Omega \) is denoted by \( \omega_{ij} \), a \( MN \times MN \) symmetric matrix with nonnegative entries. Then \( \hat{\omega}; \Omega \to \mathbb{R} \) is given by

\[
\hat{\omega}_i = \sum_{j \in \Omega} \omega_{ij}.
\]
For an image \( u \) we define its \( L^2 \)-norm by
\[
\|u\| = \left( \sum_i u_i^2 \omega_i \right)^{\frac{1}{2}}.
\]

For a pair of \( u, v : \Omega \mathbb{R} \) we define a scalar product by
\[
\langle u, v \rangle_{L^2} = \sum_i u_i v_i \omega_i.
\]

Now we define the gradient vector field of \( u \), \( du : \Omega \times \Omega \rightarrow \mathbb{R} \) by
\[
(du)_{ij} = u_j - u_i, \quad (7)
\]

This definition is different from the nonlocal gradient vector field introduced by Gilboa-Osher in [10]. As in [10] a map from \( \Omega \times \Omega \rightarrow \mathbb{R} \) is considered as a vector field. For a pair of vector fields \( p, q : \Omega \times \Omega \rightarrow \mathbb{R} \) we define a scalar product
\[
\langle p, q \rangle = \sum_{i,j} p_{ij} q_{ij} \omega_{ij}.
\]

From the scalar product we have a \( L^2 \)-norm of \( p \) given by
\[
\|p\| = \left( \sum_{i,j} p_{ij}^2 \omega_{ij} \right)^{\frac{1}{2}}.
\]

Now for a vector field \( p : \Omega \times \Omega \rightarrow \mathbb{R} \), its divergence \( \text{div} \ p : \Omega \rightarrow \mathbb{R} \) is defined by
\[
(\text{div} \ p)_i := \frac{1}{\omega_i} \sum_j (p_{ji} - p_{ij}) \omega_{ij}.
\]

One can show the following

**Proposition 2.1** For any \( u : \Omega \rightarrow \mathbb{R} \) and \( p : \Omega \times \Omega \rightarrow \mathbb{R} \)
\[
\langle \nabla u, p \rangle_{L^2} = \langle u, \text{div} \ p \rangle_{L^2}.
\]

This means that \( \text{div} \) is the adjoint operator of \( d \). Now with the gradient operator and the divergence operator we can define the (normalized) Laplacian of function \( u \) by
\[
\Delta u := \frac{1}{2} \text{div} (du),
\]
or equivalently
\[
(\Delta u)_i := u_i - \bar{u}_i,
\]
where \( \bar{u} : \Omega \rightarrow \mathbb{R} \) is the nonlocal average of \( u \)
\[
\bar{u}_i = \frac{1}{\omega_i} \sum_j u_j \omega_{ij}.
\]
A function $u$ satisfying $\Delta u = 0$, or equivalent $u - \bar{u} = 0$, is called a harmonic function. A harmonic function is a critical point of the following nonlocal Dirichlet energy

$$D(u) \equiv \frac{1}{4} \|du\|^2 = \frac{1}{4} \sum_{ij} (u_j - u_i)^2 \omega_{ij}.$$  

This functional is regularization of the nonlocal means filter, which is the same as the functional obtained by Gilboa and Osher. However, our point of view is different from theirs. This difference leads us to introduce a slightly different nonlocal TV model in [1] and $H^1$ model in [2]. Here we only consider the $H^1$ model [2].

$$F(u) = D(u) + \frac{\lambda}{2} \| f - u \|_{L^2}$$

$$= \frac{1}{4} \int \int (u(y) - u(x))^2 \omega(x, y) dy dx + \frac{\lambda}{2} \int (u - f)^2 (x) \bar{\omega}(x) dx$$

$$= \frac{1}{4} \sum_{ij} (u_j - u_i)^2 \omega_{ij} + \frac{\lambda}{2} \sum_{i} (u_i - f_i)^2 \bar{\omega}_i.$$  

(8)

The difference appears only in the fidelity term, since we use a different definition of $L^2$ norm of a function. This makes (though slightly) different between Gilboa-Osher’s nonlocal $H^1$ model and our $H^1$ model. The experiment was presented in [2]. Our nonlocal $H^1$ model is more closer the nonlocal means filter than Gilboa-Osher’s nonlocal $H^1$ model. The minimizer (critical) point of $F$ satisfies

$$\Delta u = \lambda (u - f),$$  

(9)

or equivalently

$$u_i - \bar{u}_i = -\lambda (u_i - f_i).$$  

(10)

To find the minimizer $u$ we consider the follow heat equation

$$u_t = \Delta u - \lambda (u - f)$$

$$= -(u(x) - \bar{u}(x)) - \lambda (u - f)(x),$$  

(11)

which is $L^2$ the (negative) gradient flow of $F$ in the space of functions with the $L^2$ norm with respect to $\omega$.

The discrete version of (11) is

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\sum_k \omega_{ik}} \sum_l \omega_{il}(u_l^n - u_i^n) + \lambda \Delta t (f_i - u_i^n),$$

$$= (1 - \Delta t) u_i^n + \frac{\Delta t}{\sum_k \omega_{ik}} \sum_l \omega_{il} u_l^n + \lambda \Delta t (f_i - u_i^n),$$  

(12)

where $u_i^n = u_i(n \Delta t)$. The initial condition is

$$u_i^0 = f_i.$$
If $1 \geq \Delta t(1+\lambda)$, then the coefficients in equation (12) are nonnegative. Therefore we have

**Proposition 2.2** This algorithm is stable, provided that

$$1 \geq \Delta t(1+\lambda).$$

(13)

In [2] we presented experiments to show that this new nonlocal $H^1$ model is very good in denoising. We also observed in [2] that with a few iterations one can already get good results. In this paper we choose $\lambda = 0$ and $\Delta t = 1/2$. We iterate twice. For the first iteration we have

$$u^1_i = \frac{1}{2} f_i + \frac{1}{2} N(f)_i.$$ 

Then iterate it again we obtain

$$u^2_i = \frac{1}{4} f_i + \frac{1}{4} N(f)_i + \frac{1}{2} N(N(f))_i.$$ 

(14)

This is the model we want to propose in this paper. Let us rewrite it again

$$u = \frac{1}{4} f + \frac{1}{4} N(f) + \frac{1}{2} N(N(f)).$$

(15)

Here $N^2(f) := (N(N(f)))$ is the use of nonlocal means filter twice. It is surprising that our model (15) is much better than $N^2$ and also better than the nonlocal means filter for many images in our experiments.

### 3 Experiments

For experiments, the first task is how to compute the weight $\omega$. We find it by following [4]. For a given image $f : \Omega \rightarrow \mathbb{R}$ one computes the weight by using the difference of patches around each point (pixel). The patch $p_x(f)$ of size $r \times r$ around $x \in \Omega$ is given by

$$p_x(f)(t) = f(x + t), \quad \text{for } t \in [-\frac{r}{2}, \ldots, \frac{r}{2}]^2,$$

where $r$ is an odd integer, which will be 7, or 9 in our experiments. Let $d$ be the Gaussian weight Euclidean distance, ie.,

$$d(x, y) = \|p_x(f) - p_y(f)\|_a^2,$$

where $a$ is a parameter. Here $\|p_x(f) - p_y(f)\|_a$ is the discrete version of

$$\int_{\mathbb{R}^2} G_a |f(x + t) - f(y + t)|^2 dt.$$

Now the weight we will use is computed by

$$\omega(x, y) = e^{-\frac{d(x, y)}{4a^2}}.$$
In computations of the weight, a neighborhood of $11 \times 11$ pixels is used as the search window around each pixel.

Give us a noisy image $f = u_0 + v_0$, where $u_0$ is the ordinary true image and $v_0$ is a random noise (white Gaussian noise) with mean zero and standard deviation $\sigma$. In order to compare different models, we use the peak signal-to-noise ratio (PSNR) to measure images and compare noisy images and de-noised images. The peak signal-to-noise ratio (PSNR) is defined by

$$PSNR = 10 \cdot \log_{10} \left( \frac{MAX_u^2}{MSE} \right),$$

where $MAX_u$ is the maximum possible pixel value of the image $u$. $MSE$ is the mean squared error defined by

$$MSE = \frac{1}{MN} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \| u(i,j) - u_0(i,j) \|^2.$$

Here $u$ denotes the denoised image from the noisy image $f$.

In our experiments, for simplicity we choose the filtering parameter $h = \sigma$ and in the computation of the weight $\omega$ we use $r = 5$, i.e., patch size $5 \times 5$. The search window as mentioned above is of size $11 \times 11$. All images in the experiments are of size $150 \times 150$.

| Table 1. PSNR values of noisy images and denoised images |
|----------------|----------------|----------------|----------------|
| no.             | noisy image   | NLM            | NLM2           | INLM           |
| 1.1.11          | 28.1621       | 29.9632        | 28.8558        | 30.2533        |
| 1.1.12          | 28.0764       | 30.0305        | 28.8488        | 30.3350        |
| 1.5.7           | 28.1053       | 29.9604        | 28.7325        | 30.2511        |
| 1.4.01          | 28.1799       | 32.6785        | 32.1396        | 32.7734        |
| 1.4.02          | 28.1866       | 31.9004        | 31.1439        | 32.1026        |
| 1.4.03          | 28.0882       | 33.2628        | 32.6078        | 33.2722        |
| Cameraman       | 28.1733       | 31.1497        | 30.3141        | 31.1739        |
| Lena            | 28.1945       | 31.5089        | 30.2083        | 31.5718        |
| Barbara         | 28.1196       | 32.5859        | 32.0466        | 32.5925        |

4 Conclusions

In this paper we first present a discrete version of our new nonlocal $H^1$ model which was proposed recently in [1] and [2]. In the previous experiments for the new model, we found that a few iterations of the new model already receive very good results for image denoising. With this observation, in this paper we then propose an improved nonlocal means filter (15) (INLM). This INLM is as simple as the NLM and is simpler than the new nonlocal $H^1$ model proposed.
Our experiments show that this INLM provides good results for image denoising. Especially this model is much better than the method by using NLM twice. It is also better than NLM when we consider the denoising for images with higher textures.

References


[22] X. Zhang, and Tony Chan, Wavelet Inpainting by Nonlocal Total Variation


Figure 1: Denoising of the image 1.4.02 using different methods. (a) The original image $u$, (b) a noisy image $f$ with a white Gaussian noise ($\sigma = 10$) with $PSNR = 28.19$, (c) denoised image $g$ by NLM $PSNR = 31.90$, (d) NLM2 $PSNR = 31.14$ (e) INLM $PSNR = 32.10$
Figure 2: Denoising of the image Cameraman using different methods. (a) The original image \( u \), (b) a noisy image \( f \) with a white Gaussian noise \( (\sigma = 10) \) with \( PSNR = 28.17 \), (c) denoised image \( g \) by NLM \( PSNR = 31.15 \), (d) NLM2 \( PSNR = 29.31 \) (e) INLM \( PSNR = 31.17 \).
Figure 3: Denoising of the image Lena using different methods. (a) The original image $u$, (b) a noisy image $f$ with a white Gaussian noise ($\sigma = 10$) with $PSNR = 28.19$, (c) denoised image $g$ by NLM $PSNR = 31.51$, (d) by NLM2 $PSNR = 30.21$ (e) INLM $PSNR = 31.57$. 
Figure 4: Denoising of the image Barbara using different methods. (a) The original image $u$, (b) a noisy image $f$ with a white Gaussian noise ($\sigma = 10$) with $PSNR = 28.12$, (c) denoised image $g$ by NLM $PSNR = 32.59$, (d) by NLM2 $PSNR = 32.05$ (e) INLM $PSNR = 32.59$. 